Applications of approximate aggregation methods for nonlinear discrete systems.

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In ecology,

it is usual to consider different organization level: from cells to community. Processes related with each level usually evolve in different time scale and affect to different structures.

We will deal with:

- two time scale models.
- structured populations (in any sense).

Aims:

- To perform a reduction in the number of variables.
- To find out dynamical information about the original system by means of the aggregated one.

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A simple example:
Settings

Consider a population divided into

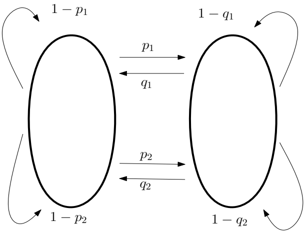
- two age classes $\Leftrightarrow$ demographic process (slow)
- two regions $\Leftrightarrow$ migratory process (fast).

State variables:

$$x_{ij}(n)$$

population density of age class i in patch j at time n.

Building up the model
Migrations



- Juvenile and adults inhabiting two regions.
- They can remain or leave its patch.

A regular stochastic matrix drives the migration process

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_1 & 0 \\ 0 & \mathcal{F}_2 \end{pmatrix} = \begin{pmatrix} 1-p_1 & q_1 & 0 & 0 \\ p_1 & 1-q_1 & 0 & 0 \\ 0 & 0 & 1-p_2 & q_2 \\ 0 & 0 & p_2 & 1-q_2 \end{pmatrix}$$

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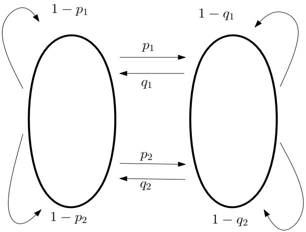

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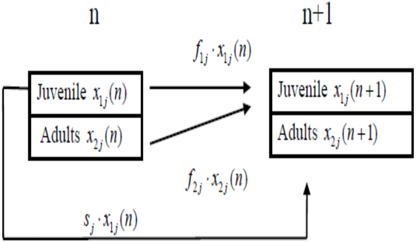
Consider local Leslie-type demography at each patch ($j = 1, 2$)



- f_{ij} fertility rate of age class i.
- s_j survival rate for juveniles.
- Local demography matrix:

$$\begin{pmatrix} f_{1j} & f_{2j} \\ s_j & 0 \end{pmatrix}$$

The demography is controlled by matrix $\mathcal{S} = \begin{pmatrix} f_{11} & 0 & f_{21} & 0 \\ 0 & f_{12} & 0 & f_{22} \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \end{pmatrix}$

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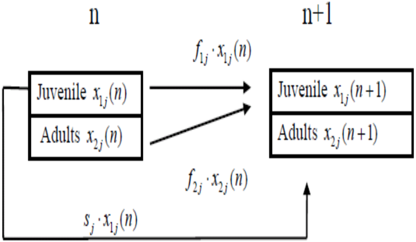

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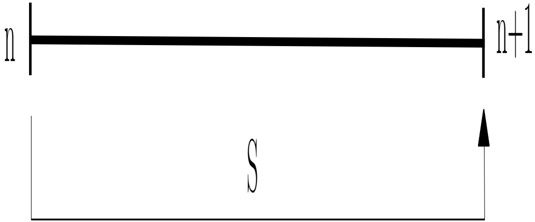
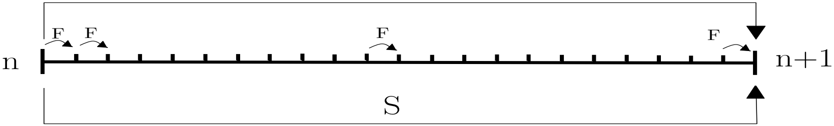
Building up the model
Time scales

Unit time *n* is that of the slow process (demography).



Building up the model
Assembling

$$F^k = F \cdot F \cdot \cdots \cdot F$$



The complete model is

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Index k stands for the ratio between the two time scales:

$$X_{k,n+1} = \mathcal{S} \cdot \mathcal{F}^k \cdot X_{k,n}$$

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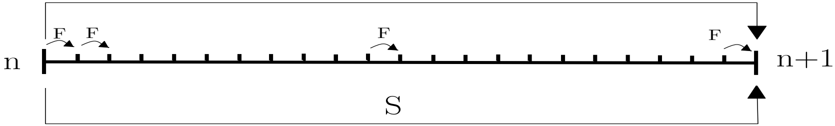

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Two time scales model
Fast equilibrium

As $\mathcal{F}$ is a regular stochastic matrix there exists $\bar{\mathcal{F}}$ such that,

$$\lim_{k\to\infty} \mathcal{F}^k X = \bar{\mathcal{F}} X \ \forall X \in \mathbb{R}^4$$

So that for k large, `general system`

$$X_{k,n+1} = \mathcal{S} \cdot \mathcal{F}^k \cdot X_{k,n}$$

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Aggregation

The Perron-Frobenius theory apply to our model.

$$\mathbf{v}_i > 0 \qquad , \qquad \mathbf{u}_i^T = (1\,1)$$

are the right and left main eigenvectors of $\mathcal{F}_i$.
The fast equilibrium decomposes as

$$\bar{\mathcal{F}} = \begin{pmatrix} \mathbf{v_1 u_1^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{v_2 u_2^T} \end{pmatrix} = \begin{pmatrix} v_1^1 & 0 \\ v_1^2 & 0 \\ 0 & v_2^1 \\ 0 & v_2^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \mathcal{T} \cdot \mathcal{G}$$

Two important properties of the migration process

1. The total density of class i keeps constant.

$$(1\ 1)\begin{pmatrix} 1-p_i & q_i \\ p_i & 1-q_i \end{pmatrix}\begin{pmatrix} x_{i1}(n) \\ x_{i2}(n) \end{pmatrix} = (1\ 1)\begin{pmatrix} x_{i1}(n) \\ x_{i2}(n) \end{pmatrix} = x_{i1}(n) + x_{i2}(n)$$

2. The migration process asymptotically distributes individuals of class i between the two patches according to the proportions established by the main right eigenvector.

$$\mathbf{v}_i = \begin{pmatrix} \dfrac{q_i}{p_i + q_i} \\ \dfrac{p_i}{p_i + q_i} \end{pmatrix} \qquad \lim_{k\to\infty} \mathcal{F}_i^k \begin{pmatrix} z_0^1 \\ z_0^2 \end{pmatrix} = (z_0^1 + z_0^2)\mathbf{v}_i$$

Vectors $\mathbf{v}_i$ are positive and normalized: $v_i^1 + v_i^2 = 1$

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We use matrix $\mathcal{G} : \mathbb{R}^4 \rightarrow \mathbb{R}^2$

1. to define the `global variables`:

$$Y(n) = \mathcal{G} \cdot X(n) = \begin{pmatrix} x_{11}(n) + x_{12}(n) \\ x_{21}(n) + x_{22}(n) \end{pmatrix}$$

which stands for the population size of each age class.

2. to get the `aggregated system` from the auxiliary system

$$X_{n+1} = S \cdot \mathcal{T} \cdot \mathcal{G} \cdot X_n$$

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Two time scales model
Following with our example

General system: $X_k(n+1) = \mathcal{S} \cdot \mathcal{F}^k \cdot X_k(n)$

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Auxiliary system: $X(n+1) = \mathcal{S} \cdot \bar{\mathcal{F}} \cdot X(n) = \mathcal{S} \cdot \mathcal{T} \cdot \mathcal{G} \cdot X(n)$

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Two time scales model

- We study the 2-dimensional aggregated system.
- The matrix $\mathcal{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ helps us to translate dynamical information of the aggregated system to the general system.
- The latest results concerning non linear discrete systems are collected in

Sanz, L; Bravo de la Parra, R; Sánchez, E.*Approximate reduction of non-linear discrete models with two time scales*.Journal of Difference Equations and Applications. 2008 1-21.

The nonlinear case

Let us consider functions

$$F, S \in \mathcal{C}^1(\Omega_N; \Omega_N),$$

where $\Omega_N \subset \mathbb{R}^N$ is an open, connected and bounded subset. The `general system` reads as

$$X_{k,n+1} = S\left(F^{(k)}(X_{k,n})\right), \qquad n = 0, 1, 2 \cdots ,$$

where

- F and S stand for the fast and the slow process
- k is the ratio between the two time scales.

General settings

Hypotheses 1: fast process reaches an equilibrium

$$\exists \bar{F} : \Omega_N \to \Omega_N, \qquad \lim_{k\to\infty} F^{(k)}(X) = \bar{F}(X), \qquad \forall\, X \in \Omega_N \subset \mathbb{R}^N.$$

from which we get the `auxiliary system`

$$X(n+1) = S\left(\bar{F}(X_n)\right).$$

Hypotheses 2: fast equilibrium decomposes in a suitable way:

$$\bar{F} = T \circ G \qquad G : \Omega_N \to \Omega_q \qquad \Omega_q \subset \mathbb{R}^q,\ q \ll N \qquad T : \Omega_q \to \Omega_N$$

Function G allow us to define the `global variables`

$$G(X) = Y \in \mathbb{R}^q,\ q \ll N$$

and the `aggregated system`

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Comparison for a fixed n

Aggregated system vs Auxiliary system

- Functions G and T exactly relate both systems
- It is called `perfect aggregation`

	Aggregated system	Auxiliary system
Initial value:	$Y_0 = G(X_0)$	X_0
Solution:	Y_n	X_n
Connection:	$Y_n = G(X_n)$	$X_n = T(Y_{n-1})$

Comparison for a fixed n

Aggregated system vs General system

If $\{X_0, \cdots, X_i = T(Y_{i-1});\ i = 1, \cdots, L\} \subset \Omega$ bounded open domain such that

$$\lim_{k\to\infty} F^{(k)} = \bar{F} \text{ uniformly on } \bar{\Omega} \subset \Omega_N,$$

we get `approximate agregation`

	Aggregated system	General system
Initial value:	$Y_0 = G(X_0)$	X_0
Solution:	Y_n	$X_{k,n}$
Connection:	$Y_n = \lim_{k\to\infty} G(X_{k,n})$	$X_{k,n} = \lim_{k\to\infty} T(Y_{n-1})$

For $n = 0, 1, 2, \cdots, L$

Asymptotic behavior

Hypothesis

There exists $\bar{F} \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\lim_{k\to\infty} F^{(k)} = \bar{F}$, $\lim_{k\to\infty} DF^{(k)} = D\bar{F}$ uniformly on $K \subset \mathbb{R}^N$ compact

Equilibrium

If Y^* is an equilibrium to the aggregated system.

- $X^* = T(Y^*)$ is an equilibrium to the auxiliaty system.
- The general system posses equilibriums X_k^* $\forall\, k \geq k_0$.
- $\lim_{k\to\infty} X_k^* = X^*$.
- If Y^* is hyperbolic and asymptotically stable (unstable), so are X^* and X_k^*.

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Asymptotic behavior

Hipotheses

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Basins of attraction

Let Y^* be hyperbolic and asymptotically stable.

Let Y_0 be such that $Y_n \longrightarrow Y^*$.

Then, the solution $X_{k,n}$ relative to $X_0 = T(Y_0)$

$$\lim_{n\to\infty} X_{k,n} = X_k^*.$$

Asymptotic behavior ($n \nearrow \infty$)

Moreover

Previous results hold for m-cycles instead of equilibrium points.

APPLICATIONS:

- Multi-attractors.
- Recovering classic models.
- Allee effect.
- Host-parasitoid interaction.

Fast dynamics depending on global variables

Results

In [2] it is shown that when $\mathcal{F}(y)$ is a regular stochastic matrix depending on global variables, there exists $\bar{\mathcal{F}} \in \mathcal{C}^1(\mathbb{R}^N)$ such that limits

$$\lim_{k\to\infty} \mathcal{F}^{(k)}(y) = \bar{\mathcal{F}}(y), \qquad \lim_{k\to\infty} D\mathcal{F}^{(k)}(y) = D\bar{\mathcal{F}}(y)$$

are uniform on compact sets $K \subset \mathbb{R}^N$.

Next, we will see some applications.

[2] Marvá, M; Sánchez E; Bravo de la Parra, R; Sanz, L; *Reduction of Slow-Fast Discrete Models Coupling Migration and Demography.* (2008) Journal of Theoretical Biology. **doi:10.1016/j.jtbi.2008.07.014**.

Multi-attractors

Fast demography versus migrations

Two age classes population inhabiting two patches

Fast Leslie-type local demography depending on global variables

$$\forall y \in \mathbb{R}_+ \ , \ \ \mathcal{F}_i(y) := \begin{pmatrix} 0 & f_{12}^i(y) \\ s_{21}^i(y) & s_{22}^i(y) \end{pmatrix} , \ i = 1, 2$$

The next condition is needed to achieve the fast equilibrium

$$s_{22}^i(y) + f_{12}^i s_{21}^i(y) = 1, \quad i = 1, 2,$$

Multi-attractors

Fast demography versus migrations

Two age classes population inhabiting two patches

We allow migrations between patches. The correponding matrix is

$$\mathcal{S} := \begin{pmatrix} 1-a_1 & 0 & a_2 & 0 \\ 0 & 1-b_1 & 0 & b_2 \\ a_1 & 0 & 1-a_2 & 0 \\ 0 & b_1 & 0 & 1-b_2 \end{pmatrix} \quad , \quad a_i, b_i \in (0,1)\,,\ i = 1,2$$

where a_i, b_i= migration rates for juveniles and adults at patch i.

Multi-attractors

Fast demography versus migrations

Global variables

$$Y_n := G(X_n) = \begin{pmatrix} x_n^{11} + (1/t_{21}^1)x_n^{12} \\ x_n^{21} + (1/t_{21}^2)x_n^{22} \end{pmatrix} := \begin{pmatrix} y_n^1 \\ y_n^2 \end{pmatrix}$$

stand for population at each path weighted by its reproductive values.

After aggregation process:

$$\left\{ \begin{array}{lcl} y_{n+1}^1 & = & \left[u_1^1(1-a_1)v_1^1(y_n^1) + u_2^1(1-b_1)v_2^1(y_n^1)\right] y_n^1 + \left[u_1^1 a_2 v_1^2(y_n^2) + u_2^1 b_2 v_2^2(y_n^2)\right] y_n^2 \\ y_{n+1}^2 & = & \left[u_1^2 a_1 v_1^1(y_n^1) + u_2^2 b_1 v_2^1(y_n^1)\right] y_n^1 + \left[u_1^2(1-a_2)v_1^2(y_n^2) + u_2^2(1-b_2)v_2^2(y_n^2)\right] y_n^2 \end{array} \right.$$

Multi-attractors

Fast demography versus migrations

Global variables

$$Y_n := G(X_n) = \begin{pmatrix} x_n^{11} + (1/t_{21}^1)x_n^{12} \\ x_n^{21} + (1/t_{21}^2)x_n^{22} \end{pmatrix} := \begin{pmatrix} y_n^1 \\ y_n^2 \end{pmatrix}$$

stand for population at each path weighted by its reproductive values.

After aggregation process:

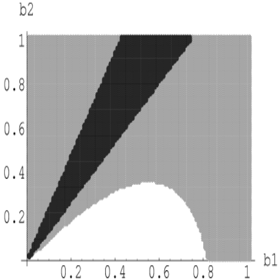
$$\begin{cases} y_{n+1}^1 = \left[u_1^1(1-a_1)v_1^1(y_n^1) + u_2^1(1-b_1)v_2^1(y_n^1)\right] y_n^1 + \left[u_1^1 a_2 v_1^2(y_n^2) + u_2^1 b_2 v_2^2(y_n^2)\right] y_n^2 \\ y_{n+1}^2 = \left[u_1^2 a_1 v_1^1(y_n^1) + u_2^2 b_1 v_2^1(y_n^1)\right] y_n^1 + \left[u_1^2(1-a_2)v_1^2(y_n^2) + u_2^2(1-b_2)v_2^2(y_n^2)\right] y_n^2 \end{cases}$$

Multi-attractors

Fast demography versus migrations

Numerical results

From white (none) to black (two): number of asymptotically stable positive equilibriums depending on the values of b_1, b_2



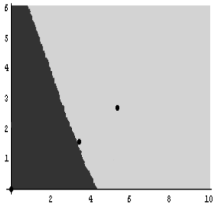
Parameter values:
$a_1 = 0.1,\ a_2 = 0.3,$
$\alpha_1 = 100,\ \alpha_2 = 45,$
$t^1_{21} = 0.3,\ t^2_{21} = 0.1$
$b_1,\ b_2 \in [0.01; 1.0],$
step 0.005.

Multi-attractors

Fast demography versus migrations

Numerical results

Basins of attraction of each equilibrium



Parameter values:

$a_1 = 0.1,\ a_2 = 0.3,$
$b_1 = 0.3,\ b_2 = 0.7,$
$\alpha_1 = 100,\ \alpha_2 = 45,$
$t_{21}^1 = 0$

Fast migrations depending on global variables vs demography

Let us consider a population

- divided into two patches.
- fast migrations depending on global variables:

$$\mathcal{F}(y) = \begin{pmatrix} 1 - a(y) & b(y) \\ a(y) & 1 - b(y) \end{pmatrix}$$

- Local demography (slow):

$$S(\mathbf{X}(n)) = (s_1(x_1(n)), s_2(x_2(n)))$$

- The complete model is

$$\mathbf{X}_k(n+1) = S\left(\mathcal{F}^k(y)\mathbf{X}_k(n)\right)$$

Fast migrations vs demography

The fast dynamics equilibrium and global variables read as

$$\bar{\mathcal{F}}(y) = \lim \mathcal{F}^k(y) = \begin{pmatrix} \frac{b(y)}{a(y)+b(y)} & \frac{b(y)}{a(y)+b(y)} \\ \frac{a(y)}{a(y)+b(y)} & \frac{a(y)}{a(y)+b(y)} \end{pmatrix} = \begin{pmatrix} \frac{b(y)}{a(y)+b(y)} \\ \frac{a(y)}{a(y)+b(y)} \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The global variables are

$$y_n = x_n^1 + x_n^2$$

And the aggregated system is

$$y_{n+1} = s_1 \left(\frac{b(y_n) y(n)}{a(y_n) + b(y_n)} \right) + s_2 \left(\frac{a(y_n) y(n)}{a(y_n) + b(y_n)} \right)$$

Recovering classic models

Fast migrations vs demography

When the local demographic is a malthusian source-sink process:

$$\mathcal{S} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \quad , \quad 0 < d_1 < 1 < d_2$$

the aggregated system becomes

$$y_{n+1} = \left(\frac{d_1 b(y_n) + d_2 a(y_n)}{a(y_n) + b(y_n)} \right) y_n$$

Recovering classic models

Fast migrations vs demography

With appropriate migration rates we recover classical models

Model	Migration rates
Beverton-Holt $y_{n+1} = \dfrac{\alpha y_n}{1+\beta y_n}$	$a(y) := \frac{\alpha - d_1(1+\beta y)}{d_2 - d_1}$; $b(y) := \frac{d_2(1+\beta y)-\alpha}{d_2-d_1}$
Ricker $y_{n+1} = e^{r(1-y_n/K)} y_n$	$a(y) := \frac{e^{r(1-y/K)} - d_1}{d_2-d_1}$; $b(y) := \frac{d_2 - e^{r(1-y/K)}}{d_2-d_1}$

Recovering classic models
Fast migrations vs demography

With appropriate migration rates we recover classical models

Model	Migration rates
Beverton-Holt $y_{n+1} = \dfrac{\alpha y_n}{1+\beta y_n}$	$a(y) := \frac{\alpha - d_1(1+\beta y)}{d_2 - d_1}$; $b(y) := \frac{d_2(1+\beta y) - \alpha}{d_2 - d_1}$
Ricker $y_{n+1} = e^{r(1-y_n/K)} y_n$	$a(y) := \frac{e^{r(1-y/K)} - d_1}{d_2 - d_1}$; $b(y) := \frac{d_2 - e^{r(1-y/K)}}{d_2 - d_1}$

Allee effect

Fast migrations vs demography

Source-sink local demography	Migration rates
Malthusian $x_{i,n+1} = s_i(x_{i,n}) = d_i x_{i,n}$ $0 < d_1 < 1 < d_2$	$a(y) := \frac{y^2}{y^2+\beta} \quad ; \quad b(y) := \frac{y^2+\beta}{y^2+\delta}$ $0 < \beta < \delta$
Beverton-Holt $x_{i,n+1} = s_i(x_{i,n}) = \frac{d_i x_{i,n}}{1 + c_i x_{i,n}}$ $0 < d_1 < 1 < d_2, \quad 0 < c_i$	$a(y) := \frac{y}{1+y} \quad ; \quad b(y) := \frac{1}{1+y}$

The corresponding aggregated systems ate

Malthus
$$y_{n+1} = \left(\frac{d_1 \left(y^2+\beta\right)^2 + d_2 \left(y^2+\delta\right) y^2}{\left(y^2+\beta\right)^2 + \left(y^2+\delta\right) y^2} \right) y_n$$

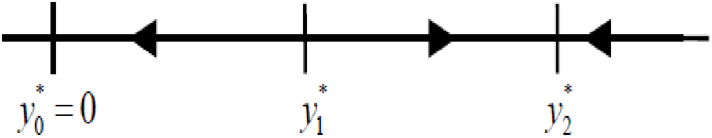
Beverton-Holt
$$y_{n+1} = \left(\frac{d_1}{1+(1+c_1)y_n} + \frac{d_2 y_n}{1+y_n+c_2 y_n^2} \right) y_n$$

Allee effect
Fast migrations vs demography

There exist parameter values such that the aggregated system posses 3 equilibriums

$$y_0^* = 0 < y_1^* < y_2^*$$

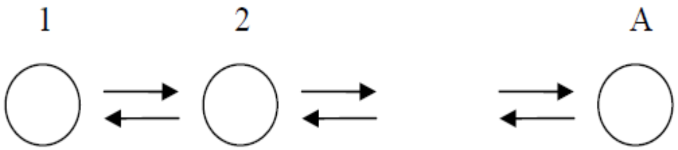
where y_0^*, y_2^* are sinks and y_1^* is a source and so, a persistence population threshold:



Host-Parasitoid in a chained environment

Fast migrations vs demography

Let us suppose *A* chained patches



Individuals can move toward adjacent patches.

Host-Parasitoid in a chained environment

Fast migrations vs demography

At patch i

1. Fast migrations:

$$\begin{pmatrix} H_{i,n+1} \\ P_{i,n+1} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 & 0 \\ 0 & \mathcal{F}_2(H_{i,n}) \end{pmatrix} \begin{pmatrix} H_{i,n} \\ P_{i,n} \end{pmatrix}$$

   where $\mathcal{F}_j$ are regular and stochastic matrices.

2. Local host-parasitoid Nicholson-Bayley slow interaction:

$$\begin{cases} H_{i,n+1} = \lambda_i H_{i,n} e^{-a_i P_{i,n}} \\ P_{i,n+1} = C_i H_{i,n}(1 - e^{-a_i P_{i,n}}) \end{cases}$$

Host-Parasitoid in a chained environment

Fast migrations vs demography

At patch i

1. Fast migrations:

$$\begin{pmatrix} H_{i,n+1} \\ P_{i,n+1} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 & 0 \\ 0 & \mathcal{F}_2(H_{i,n}) \end{pmatrix} \begin{pmatrix} H_{i,n} \\ P_{i,n} \end{pmatrix}$$

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Host-Parasitoid in a chained environment

Hosts migration matrix

$$\mathcal{F}_1 = \begin{pmatrix} 1-\alpha f & f & 0 & \cdots & \cdots & 0 \\ \alpha f & 1-(1+\alpha)f & f & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \alpha f & 1-(1+\alpha)f & f \\ 0 & \cdots & \cdots & 0 & \alpha f & 1-f \end{pmatrix}$$

Parameter α weights migrations toward left or right.

Host-Parasitoid in a chained environment

Parasitoid migration matrix

$$\mathcal{F}_2(H_n) = \begin{pmatrix} 1-\frac{g(h_{1,n})}{2} & \frac{g(h_{2,n})}{2} & 0 & \cdots & \cdots & 0 \\ \frac{g(h_{1,n})}{2} & 1-g(h_{2,n}) & \frac{g(h_{3,n})}{2} & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \frac{g(h_{A-2,n})}{2} & 1-g(h_{A-1,n}) & \frac{g(h_{A,n})}{2} \\ 0 & \cdots & \cdots & 0 & \frac{g(h_{A-1,n})}{2} & 1-\frac{g(h_{A,n})}{2} \end{pmatrix}$$

Where

$$g(h_{i,n}) = \frac{1}{1+h_{i,n}^{\beta}}$$

Fast migrations depending on some state variables

Results [3]

Given system

$$\begin{pmatrix} X^1_{k,n+1} \\ X^2_{k,n+1} \end{pmatrix} = \mathcal{S} \circ \left(\begin{pmatrix} \mathcal{F}^k_1 & 0 \\ 0 & \mathcal{F}^{(k)}_2(X^1_{k,n}) \end{pmatrix} \cdot \begin{pmatrix} X^1_{k,n} \\ X^2_{k,n} \end{pmatrix} \right)$$

if $\mathcal{F}_i$ are stochastic and regular matrices, then $\exists \bar{\mathcal{F}} \in \mathcal{C}^1(\mathbb{R}^N)$;

$$\lim_{k\to\infty} \mathcal{F}^{(k)}(X^1)X = \bar{\mathcal{F}}(X^1)X, \qquad \lim_{k\to\infty} D\mathcal{F}^{(k)}(X^1)X = D\bar{\mathcal{F}}(X^1)X$$

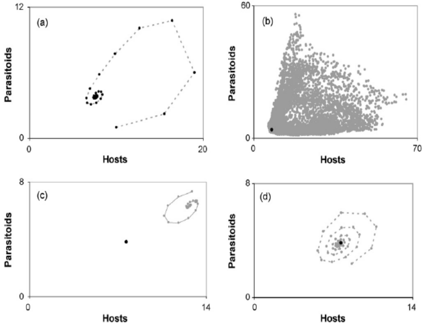
$X \in \Omega_N$ are uniform on compact sets $K \subset \mathbb{R}^N$.

[3] Tri Nguyen Huu, Pierre Auger, Christophe Lett, Marcos Marvá. *Emergence of global behaviour in a host-parasitoid model with density-dependent dispersal in a chain of patches.* Ecological Complexity 5 (2008), 9-21.

Host-Parasitoid in a chained environment

Fast migrations vs demography

For certain parameter values the aggregated system has a positive asymptotically stable equilibrium

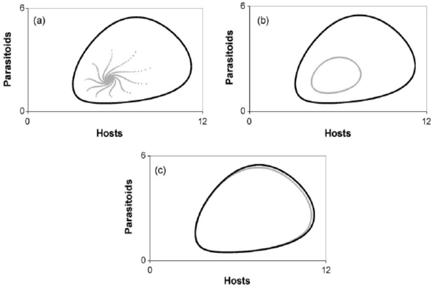


$A = 5,\ \alpha = 0.5,\ \beta = 4,$
$f = 0.2,\ \lambda_i = 2,\ a_i = 0.5,\ c_i = 1,$
$i \in [1, 2, \cdots, 5].$
(a) Aggregated system.
(b), (c), (d) general system for
$k = 2, 4, 50.$

Host-Parasitoid in a chained environment

Fast migrations vs demography

Numerical experiments strongly suggest that the existence of invariant curves to the aggregated system entail that to the general system.



$A = 5$, $\alpha = 2.3$, $\beta = 3$, $f = 0.2$, $\lambda_i = 2$, $a_i = 0.5$, $c_i = 1$, $i \in [1, 2, \cdots, 5]$.
Black: aggregated system.
Grey: general system (a), (b), (c) for $k = 2, 4, 50$.

We work in theoretical results thit support that relation.