# APPROXIMATE AGGREGATION OF NON-AUTONOMOUS TWO TIME SCALES SPATIALLY DISTRIBUTED SYSTEMS.

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- Reduction of the dimension.
- Non autonomous systems.
- Different ways of introducing time dependence.

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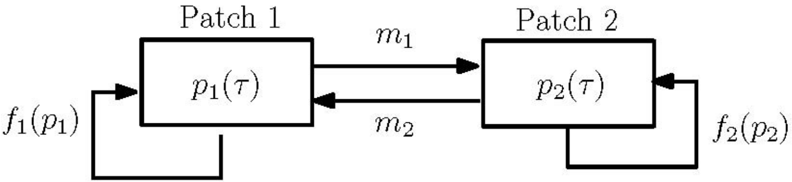
# Migrations and local interactions




Migrations

$$\begin{cases} \frac{dp_1}{d\tau} = -m_1 p_1 + m_2 p_2 \\ \frac{dp_2}{d\tau} = m_1 p_1 - m_2 p_2 \end{cases}$$

Local dynamics

$$\begin{cases} \frac{dp_1}{d\tau} = f_1(p_1) \\ \frac{dp_2}{d\tau} = f_2(p_2) \end{cases}$$

# Fast migrations - Slow local interactions

Two time scales model:

$$\begin{cases} \dfrac{dp_1}{d\tau} = -m_1 p_1 + m_2 p_2 + \epsilon f_1(p_1) \\ \\ \dfrac{dp_2}{d\tau} = m_1 p_1 - m_2 p_2 + \epsilon f_2(p_2) \end{cases} \tag{1}$$

- Slow terms are in front of $\epsilon$.
- Time dependence on $t = \epsilon\tau$

$$\begin{cases} \dfrac{dp_1}{d\tau} = -m_1(\epsilon\tau)p_1 + m_2(\epsilon\tau)p_2 + \epsilon f_1((\epsilon\tau), p_1) \\ \\ \dfrac{dp_2}{d\tau} = m_1(\epsilon\tau)p_1 - m_2(\epsilon\tau)p_2 + \epsilon f_2((\epsilon\tau), p_2) \end{cases} \tag{2}$$

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# Slow-fast form

- Slow variable $z$ varies slowly: $dz/d\tau = \mathcal{O}(\epsilon)$

- Defining $p = p_1 + p_2$ and frequencies $\nu_i = p_i/p$ yields

$$\begin{cases} \dfrac{d\nu_1}{d\tau} = m_2(\epsilon\tau) - (m_1(\epsilon\tau) + m_2(\epsilon\tau))\nu_1 + \epsilon \dfrac{f_1((\epsilon\tau), \nu_1 p)}{p} \\ \dfrac{dp}{d\tau} = \epsilon\,(f_1((\epsilon\tau), \nu_1 p) + f_2((\epsilon\tau), (1-\nu_1)p)) \end{cases} \tag{3}$$

Change variables $s = (\epsilon\tau - \alpha)/\epsilon$ and let $\epsilon \to 0$ in (3):

$$\begin{cases} \dfrac{d\nu_1}{ds} = m_2(\alpha) - (m_1(\alpha) + m_2(\alpha))\nu_1 \\ \\ \dfrac{dp}{ds} = 0 \end{cases}$$

For each $\alpha$

$$\nu_1^*(\alpha) = \frac{m_2(\alpha)}{m_1(\alpha) + m_2(\alpha)}$$

is an A.S. equilibrium, uniformly in $\alpha$.

# Aggregation result

Let $t = \epsilon\tau$ and consider $\nu_1^*(t) = \dfrac{m_2(t)}{m_1(t) + m_2(t)}$

**Theorem**

*If an uniformly AS solution $p^*(t)$ exits for equation*

$$\frac{dp}{dt} = f_1(t, \nu_1^* p) + f_2(t, (1 - \nu_1^*)p)$$

*then, for each solution $(p_1^\epsilon(t), p_2^\epsilon(t))$ of system (2),*

$$\lim_{\epsilon \to 0} (p_1^\epsilon(t), p_2^\epsilon(t)) = (\nu_1^*(t)p^*(t), (1 - \nu_1^*(t))p^*(t))$$

*uniformly in closed subintervals of $[t_0, \infty)$*

where $m_i$, $f_i \in \mathcal{C}^2$ are periodic functions of time, $i = 1, 2$.

# Fast dynamics depending on $\tau$

When

$$\begin{cases} \dfrac{dp_1}{d\tau} = -m_1(\tau)p_1 + m_2(\tau)p_2 + \epsilon f_1((\epsilon\tau), p_1) \\[2ex] \dfrac{dp_2}{d\tau} = m_1(\tau)p_1 - m_2(\tau)p_2 + \epsilon f_2((\epsilon\tau), p_2) \end{cases}$$

when changing variable $s = (\epsilon\tau - \alpha)/\epsilon$ and letting $\epsilon \to 0$, it must exists $\bar{m}_i := \lim_{\epsilon \to 0} m_i(s + \alpha/\epsilon)$

$$\begin{cases} \dfrac{d\nu_1}{ds} = \bar{m}_2 - (\bar{m}_1 + \bar{m}_2)\nu_1 \\[2ex] \dfrac{dp}{ds} = 0 \end{cases} \tag{4}$$

has an A.S. equilibrium $\nu_1^* = \dfrac{\bar{m}_2}{\bar{m}_1 + \bar{m}_2}$.

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# General settings

Consider a population divided in $A$ groups inhabiting $q$ patches:

$$\mathbf{n} = (\mathbf{n}_1, \cdots, \mathbf{n}_A) = (n_{11}, \cdots, n_{1q}, \cdots, n_{A1}, \cdots, n_{Aq}) \in \mathbb{R}^{qA}$$

Our model couples fast migrations between patches with slow local interactions:

$$\begin{cases} \dfrac{d\mathbf{n}_1}{d\tau} = \mathbf{f}_1(\star, \mathbf{n}_1) + \epsilon \mathbf{s}_1(\epsilon\tau, \mathbf{n}) \\ \qquad\cdots \\ \dfrac{d\mathbf{n}_A}{d\tau} = \mathbf{f}_A(\star, \mathbf{n}_A) + \epsilon \mathbf{s}_A(\epsilon\tau, \mathbf{n}) \end{cases} \quad \text{time } \star \in \{\tau, \epsilon\tau\} \qquad (5)$$

where

- $\mathbf{f}_i, \mathbf{s}_i \in \mathcal{C}^2$.
- $\mathbf{f}_i$ migrations of group $i$.
- $\mathbf{s}_i$ interactions of group $i$ with other(s) groups.

# Slow-fast form

We consider:

- the global (slow) variables

$$y_i := \sum_{k=1}^{q} n_{ik} \qquad i = 1, \cdots, A$$

- the frequencies

$$x_{ij} = \frac{n_{ij}}{y_i} \qquad i = 1, \cdots, A, \; j = 1, \cdots, q.$$

Rearranging variables $\mathbf{n} \in \mathbb{R}^{qA} \leftrightarrow (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{(q-1)A} \times \mathbb{R}^{A}$

$$\begin{cases} \dfrac{d\mathbf{x}}{d\tau} = \mathbf{F}(\star, \mathbf{x}, \mathbf{y}) + \epsilon \mathbf{S}(\star, \mathbf{x}, \mathbf{y}), & \mathbf{x} \in \mathbb{R}^{(q-1)A} \\ \dfrac{d\mathbf{y}}{d\tau} = \epsilon \mathbf{G}_j(\star, \mathbf{x}, \mathbf{y}), & \mathbf{y} \in \mathbb{R}^{A}. \end{cases} \tag{6}$$

# Slow-fast form

If migrations are non autonomous and linear:

$$\mathbf{f}_j(\star, \mathbf{n}_j) = \begin{pmatrix} -m_{1j}(\star) & \alpha^j_{12} m_{2j}(\star) & \cdots & \alpha^j_{1q} m_{qj}(\star) \\ \alpha^j_{21} m_{1j}(\star) & -m_{2j}(\star) & \cdots & \alpha^j_{2q} m_{qj}(\star) \\ \dots & \dots & \dots & \dots \\ \alpha^j_{q1} m_{1j}(\star) & \alpha^j_{q2} m_{2j}(\star) & \cdots & -m_{qj}(\star) \end{pmatrix} \begin{pmatrix} n_{1j} \\ n_{2j} \\ \dots \\ n_{qj} \end{pmatrix}$$

where $\star \in \{\tau, \epsilon\tau\}$, $0 \le \alpha^j_{mk} \le 1$, $\sum_{k \neq m, k=1}^{q} \alpha^j_{mk} = 1$,

$m = 1, 2, \cdots, q$, then

# Fast dynamics depending on $t = \epsilon\tau$

**Theorem.** Under the previous hypotheses, system (6) becomes

$$\begin{cases} \mathbf{x}'_t = \mathbf{F}(t,\mathbf{x}) + \epsilon\mathbf{S}(t,\mathbf{x},\mathbf{y}), & \mathbf{x} \in \mathbb{R}^{(q-1)A} \\ \mathbf{y}'_t = \mathbf{G}(t,\mathbf{x},\mathbf{y}), & \mathbf{y} \in \mathbb{R}^{A} \end{cases}$$

where $\mathbf{F}, \mathbf{G}, \mathbf{S} \in \mathcal{C}^2$ are $\omega$-periodic on $t$. Assume that system

$$\mathbf{y} = \mathbf{G}(t, \mathbf{x}^*(t), \mathbf{y})$$

has an uniformly A.S. solution $\mathbf{y}^*(t)$. Then, for each $\epsilon > 0$ small enough, the corresponding solution $(\mathbf{n}_1^\epsilon(t), \cdots, \mathbf{n}_A^\epsilon(t))$ of the original system (5) verifies

$$\lim_{\epsilon\to 0}(\mathbf{n}_1^\epsilon(t), ..., \mathbf{n}_A^\epsilon(t)) =$$

$$\left(\mathbf{x}_1^* y_1^*, \left(1 - \sum_{k=1}^{q-1} x_{1k}^*\right) y_1^*, ..., \mathbf{x}_A^* y_A^*, \left(1 - \sum_{k=1}^{q-1} x_{Ak}^*\right) y_A^*\right)$$

uniformly on closed subset of $[t_0, \infty)$.

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uniformly on closed subset of $[t_0,\infty)$.

# Fast dynamics depending on $\tau$

**Theorem.** Consider the system

$$\begin{cases} \mathbf{x}_t' = \mathbf{F}(\tau, \mathbf{x}) + \epsilon \mathbf{S}(t, \mathbf{x}, \mathbf{y}), & \mathbf{x} \in \mathbb{R}^{(q-1)A} \\ \mathbf{y}_t' = \mathbf{G}(t, \mathbf{x}, \mathbf{y}), & \mathbf{y} \in \mathbb{R}^{A} \end{cases}$$

where $\mathbf{S}, \mathbf{G}$ are $\omega$-periodic on $t$ and $\lim_{\xi \to \infty} \mathbf{F}(\xi, \mathbf{x}) = \bar{\mathbf{F}}(\mathbf{x})$. Assume that system

$$\mathbf{y} = \mathbf{G}(t, \mathbf{x}^*, \mathbf{y})$$

has an A.S. solution $\mathbf{y}^*(t)$. Then, for each $\epsilon > 0$ small enough, the corresponding solution $(\mathbf{x}_\epsilon(t), \mathbf{y}_\epsilon(t))$ of the original system (6) verifies

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uniformly on closed subset of $[t_0, \infty)$.

For more general migration functions (for instance, a matrix with periodic and denso-dependent coefficients,...)

$$\begin{cases} \dfrac{d\mathbf{x}}{ds} = \mathbf{F}(\alpha, \mathbf{x}, \mathbf{y}) & \mathbf{x} \in \mathbb{R}^{(q-1)A} \\ \\ \dfrac{d\mathbf{y}}{ds} = 0 & \mathbf{y} \in \mathbb{R}^{A}. \end{cases}$$

it is needed that system

$$\mathbf{x}'_s = \mathbf{F}(\alpha, \mathbf{x}, \beta), \qquad (\alpha, \mathbf{x}, \beta) \in \mathbb{R} \times \mathbb{R}^{(q-1)A} \times \mathbb{R}^q$$

possess finitely many equilibrium $\mathbf{x}^*(\alpha, \beta)$ isolated and A.S. uniformly in $(\alpha, \beta)$.

# Prey-predator system with refuge

$$\begin{cases} \dfrac{dn_1}{d\tau} = -m_1(t)n_1 + m_2(t)n_2 \\ \\ \dfrac{dn_2}{d\tau} = m_1(t)n_1 - m_2(t)n_2 \\ \\ \\ \end{cases}$$

Coefficients are positive periodic functions of time $t = \epsilon\tau$.

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$$\begin{cases} \dfrac{dn_1}{d\tau} = -m_1(t)n_1 + m_2(t)n_2 + \epsilon\left[\lambda_1(t)n_1\left(1 - \dfrac{n_1}{k_1(t)}\right)\right] \\ \\ \dfrac{dn_2}{d\tau} = m_1(t)n_1 - m_2(t)n_2 \\ \\ \end{cases}$$

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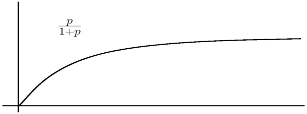

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Coefficients are positive periodic functions of time.

Consider:

- global variables: $n = n_1 + n_2$, $p$.
- frequencies: $\nu_i = \dfrac{n_i}{n}$, $1 = \nu_1(t) + \nu_2(t)$, $\nu_1^*(t) = \frac{\nu_1(t)}{\nu_1(t)+\nu_2(t)}$.

# The aggregated system

Proceeding as before and rearranging terms we get the aggregated system

$$\begin{cases} n'_t = \left(a(t) - b(t)n - \dfrac{c(t)\,p}{1+p}\right) n \\ p'_t = \left(-\lambda(t) + \dfrac{f(t)\,n}{1+p}\right) p \end{cases} \tag{7}$$

with $\omega$-periodic positive coefficients. Where

$$a(t) = \lambda_1(t)\nu_1^*(t) + \lambda_2(t)(1-\nu_1^*) \qquad b(t) = \frac{\lambda_1(t)(\nu_1^*)^2}{k_1(t)} + \frac{\lambda_2(t)(1-\nu_1^*)^2}{k_2(t)}$$

$$c(t) = \beta_2(t)(1-\nu_1^*) \qquad \lambda(t) = \lambda_3(t) \qquad f(t) = \beta_3(t)\nu_1^*$$

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# Looking for uniformly asymptotically stable solutions

The aggregated system (7) posses

- The trivial solution: $(n(t), p(t)) = (0, 0)$
- Solving

$$p'_t = -\lambda_3(t)p \qquad \text{and} \qquad n'_t = (a(t) - b(t)n)n$$

  yield the semi-trivial solutions:
  - $(n(t), p(t)) = (0, p^*(t))$.
  - $(n(t), p(t)) = (n^*(t), 0)$ where $n^*(t)$ is the periodic globally asymptotically stable solution of the periodic logistic equation with positive coefficients.
- Positive solutions.

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# Existence of periodic positive solutions

Using the maximum and minimum of the coefficients we get

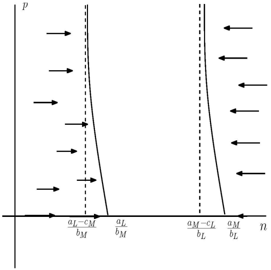
$$\begin{cases} \left(a_L - b_M n - c_M \dfrac{p}{1+p}\right) n \le n'_t \le \left(a_M - b_L n - c_L \dfrac{p}{1+p}\right) n \\ \left(-\lambda_M + f_L \dfrac{n}{1+p}\right) p \le p'_t \le \left(-\lambda_L + f_M \dfrac{n}{1+p}\right) p \end{cases} \tag{8}$$

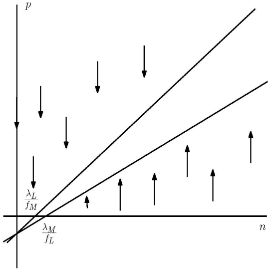
## Proposition

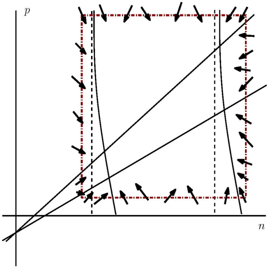
*Every solution of the aggregated system* (7) *is uniformly bounded, i.e.*

$$\lim_{t\to\infty} (n(t), p(t)) \in \left[0, \frac{a_M}{b_L}\right] \times \left[0, \frac{f_M a_M}{\lambda_L b_L} - 1\right]$$

# Existence of periodic positive solutions




Invariant region $R$. $0 < \frac{\lambda_M}{f_L} < \frac{a_L - c_M}{b_M}$.

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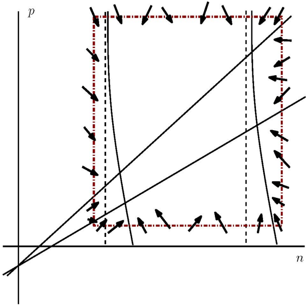



Invariant region $R$. $0 < \frac{\lambda_M}{f_L} < \frac{a_L - c_M}{b_M}$.

- We consider the $\omega$-operator

$$\begin{array}{rcl} \varphi_\omega : R & \to & R \\ (r,s) & \mapsto & \varphi_\omega(r,s) = \varphi(\omega; 0, r, s) \end{array}$$

- Brouwer's fixed point theorem: $\varphi(\omega, 0, r, s) = \varphi(0, 0, r, s)$ and $\varphi(t, 0, r, s)$ is positive, periodic and globally defined.

# Stability and uniqueness

Linearizing around the positive periodic solution $\varphi = (n_0, p_0)$

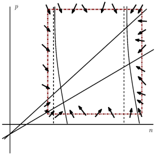
**Proposition**

*Every positive periodic solution of the aggregated system is uniformly asymptotically stable*

Using the topological degree

**Proposition**

*If the positive invariant region R is bounded away from the axes, then it contains one and only one positive periodic solution of the aggregated system.*



## Proposition

*If* $0 < \dfrac{\lambda_M}{f_L} < \dfrac{a_L - c_M}{b_M}$ *holds then, there exists a positive periodic solution* $(n_0^*(t),\, p_0^*(t))$ *of the aggregated system such that each solutions* $(n_1^\epsilon(t), n_2^\epsilon(t), p^\epsilon(t))$ *of the complete system, for* $\epsilon \sim 0$*,*

$$\lim_{\epsilon \to 0}(n_1^\epsilon(t), n_2^\epsilon(t), p^\epsilon(t)) = (\nu_1^*(t)n_0^*(t), (1-\nu_1^*(t))n_0^*(t), p_0^*(t))$$

*uniformly on closed subset of* $[t_0, \infty)$*.*
*In this case, the semi-trivial solution is unstable.*

## Remark

*The existence of* $(n_0^*(t),\, p_0^*(t))$ *depends on* $\nu_1^*(t)$ *as* $a(t),\, b(t),\, c(t),\, f(t)$ *depend on* $\nu_1^*(t)$

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# Stability of the semi-trivial solution

## Proposition

*The semi-trivial solution is uniformly asymptotically stable if*

$$\int_t^{t+T} (-\lambda(s) + f(s)n^*(s))\, ds < 0.$$

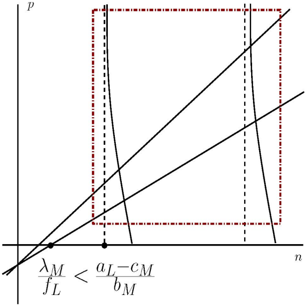
*In this case, for each solution* $(n_1^\epsilon(t), n_2^\epsilon(t), p^\epsilon(t))$ *of the complete system, for* $\epsilon \sim 0$*,*

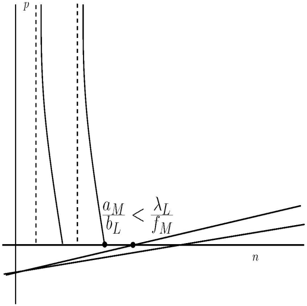
$$\lim_{\epsilon \to 0} (n_1^\epsilon(t), n_2^\epsilon(t), p^\epsilon(t)) = (\nu_1(t)n^*(t), (1-\nu_1)(t)n^*(t), 0)$$

*uniformly on closed subset of* $[t_0, \infty)$*.*

## Remark

*The stability of* $(n^*(t), 0)$ *depends on* $\nu_1^*(t)$ *as* $f(t)$ *depends on* $\nu_1^*(t)$*.*


p
n
$\frac{\lambda_M}{f_L} < \frac{a_L - c_M}{b_M}$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We can recover the behavior of the solutions of the complete model by means of uniformly asymptotically stable solutions of the aggregated system.

Fast migrations can be replaced by processes reaching an stable behavior (in a suitable way).

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Linearizing around the positive periodic solution $(n_0, p_0)$ yields

$$X' = A(t)X \tag{9}$$

where

$$A(t) = \begin{pmatrix} a(t) - 2b(t)n_0(t) - c(t)\frac{p_0(t)}{1+p_0(t)} & -c(t)\frac{n_0(t)}{(1+p_0(t))^2} \\ f(t)\frac{p_0(t)}{1+p_0(t)} & -\lambda(t) + f(t)\frac{n_0(t)}{(1+p_0(t))^2} \end{pmatrix} \tag{10}$$

changing variables $y_1 = x_1/n_0$, $y_2 = x_2/p_0$ leads to

$$Y' = B(t)Y \tag{11}$$

where

$$B(t) = \begin{pmatrix} -b(t)n_0(t) & -c(t)\dfrac{p_0(t)}{(1+p_0(t))^2} \\ f(t)\dfrac{n_0(t)}{1+p_0(t)} & -f(t)\dfrac{p_0(t)n_0(t)}{(1+p_0(t))^2} \end{pmatrix}, \tag{12}$$

which is equivalent to (10).

Consider matrix

$$\begin{pmatrix} -b_{11} & -b_{12} \\ b_{21} & -b_{22} \end{pmatrix}, \tag{13}$$

with $b_{i,j} > 0$, $i, j = 1, 2$ positive real numbers. It is clear that the eigenvalues of (13) are given by

$$\lambda = \frac{-(b_{11} + b_{22}) \pm \sqrt{(b_{11} + b_{22})^2 - 4(b_{11}b_{22} + b_{12}b_{21})}}{2} \tag{14}$$

and the real part of both eigenvalues is strictly negative.

# Uniqueness: topological degree

Consider initial values $(r_1, s_1) \in \mathrm{Int}(R)$ and the functions

$$\begin{array}{rcl} V,\ W : \mathbb{R}^2_+ & \to & \mathbb{R}^2_+ \\ (r,s) & \mapsto & V(r,s) := (r,s) - (p(T;0,r,s), z(T;0,r,s)) \\ (r,s) & \mapsto & W(r,s) := (r,s) - (r_1, s_1) \end{array}$$

$$N(r,s,\xi) := (r_1 + \xi\,[p(T,r,s) - r_1]\,;\, s_1 + \xi\,[z(T,r,s) - s_1])\,.$$

where

- $V(r,s) \neq (0,0) \neq W(r,s)$ for all $(r,s) \in \partial R$.
- $I - N$ is an admissible homotopy between $V$ and $W$, then

$$d\,[W,R,0] = d\,[V,R,0] = 1$$

The uniqueness follows from

$$|JW(\varphi)| = (1-\lambda_1)(1-\lambda_2) > 0.$$

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