TIME SCALES AND EPIDEMICS ON SMALL NETWORKS.

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Why time scales? Towns and their influence area can be partitioned in patches attending to features as healthiness or exposure to vectors. Mass public/private transport enables individuals to move between patches several times daily. These movement rates are large compared with epidemic parameters, for instance, when dealing with diseases with low transmission and recovery rates.

Summing up, models coupling two processes which evolve at (very) different rates yield two time scale systems.

How does it work? We sketch the so called quick derivation method for autonomous systems [1] which (formally) works almost the same for nonautonomous systems [2], [4]. Let $f, s : \mathbb{R}^N \to \mathbb{R}^N$ stand for the fast and the slow process. The prototype of two time scale systems reads as

$$dn/d\tau = f(n) + \varepsilon s(n) \qquad (1)$$

where parameter $\varepsilon \sim 0^+$ stands for time scales ratio. Let us change variables $n \mapsto (x, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$ in (1) where y, the slow variables, are invariant by the fast process. Changing variables in (1) yields

$$\begin{cases} dx/d\tau = F(x,y) + \varepsilon G(x,y), \\ dy/d\tau = \varepsilon S(x,y). \end{cases}$$

Let $(x^*(y), y)$ be an asymptotically stable equilibrium of $dx/d\tau = F(x,y)$ for each $y \in \mathbb{R}^k$. The so-called fast equilibrium x^* enables a sort of variables decoupling and dimension reduction. Namely, we can get certain asymptotic information of system (1) through the reduced system

$$dy/dt = S(x^*(y), y) \qquad \text{where } t = \varepsilon\tau. \qquad (2)$$

APPLICATIONS. Based in [3] we consider a population living in a N patches environment. There is a SIS-epidemic process at each patch and individuals can move between patches. At patch j $(j = 1, \cdots, N)$

- S_j, I_j stand for susceptible and infected individuals.
- β_j, γ_j stand for the infection and recovery rates,
- m^S_{ij}, m^I_{ij} stand for the susceptible and infected displacement rate from patch j to patch i.

The equations at patch j of the two time scales system read as follows

$$\begin{cases} dS_j/d\tau = -\sum_{l \neq j} m^S_{lj} S_j + \sum_{l \neq j} m^S_{jl} S_l + \epsilon \left[\dfrac{-\beta_j S_j I_j}{S_j + I_j} + \gamma_j I_j \right], \\ dI_j/d\tau = -\sum_{l \neq j} m^I_{lj} I_j + \sum_{l \neq j} m^I_{jl} I_l + \epsilon \left[\dfrac{\beta_j S_j I_j}{S_j + I_j} - \gamma_j I_j \right]. \end{cases} \qquad (3)$$

In order to reduce the model we note that

- We change variables $(S_1, I_1, \cdots, S_N, I_N) \mapsto (S, I, S_2, I_2, \cdots, S_N, I_N)$; the slow variables $I := \sum I_j$, $S := \sum S_j$ are invariant by displacements.
- Displacements in (3) are linear and can be represented through matrices M_S and M_I defined in terms of m^S_{ij} and m^I_{ij}. M_S and M_I are assumed to be irreducible. Considering just fast dynamics ($\varepsilon = 0$ in (3)) variables achieve a stable distribution among patches

$$\lim_{\tau \to \infty} (S_1(\tau), \cdots, S_N(\tau)) = (\mu_1, \cdots, \mu_N)\, S,$$
$$\lim_{\tau \to \infty} (I_1(\tau), \cdots, I_N(\tau)) = (\nu_1, \cdots, \nu_N)\, I,$$

which yields the fast equilibrium. Applying the quick derivation method to the system (3) (dimension $2N$) yields the reduced (dimension 2) system

$$\begin{cases} dS/dt = \sum_{j=1}^{N} \dfrac{-\mu_j \nu_j \beta_j S I}{\mu_j S + \nu_j I} + \sum_{j=1}^{N} \nu_j \gamma_j I, \\ dI/dt = \sum_{j=1}^{N} \dfrac{\mu_j \nu_j \beta_j S I}{\mu_j S + \nu_j I} - \sum_{j=1}^{N} \nu_j \gamma_j I, \end{cases} \qquad (4)$$

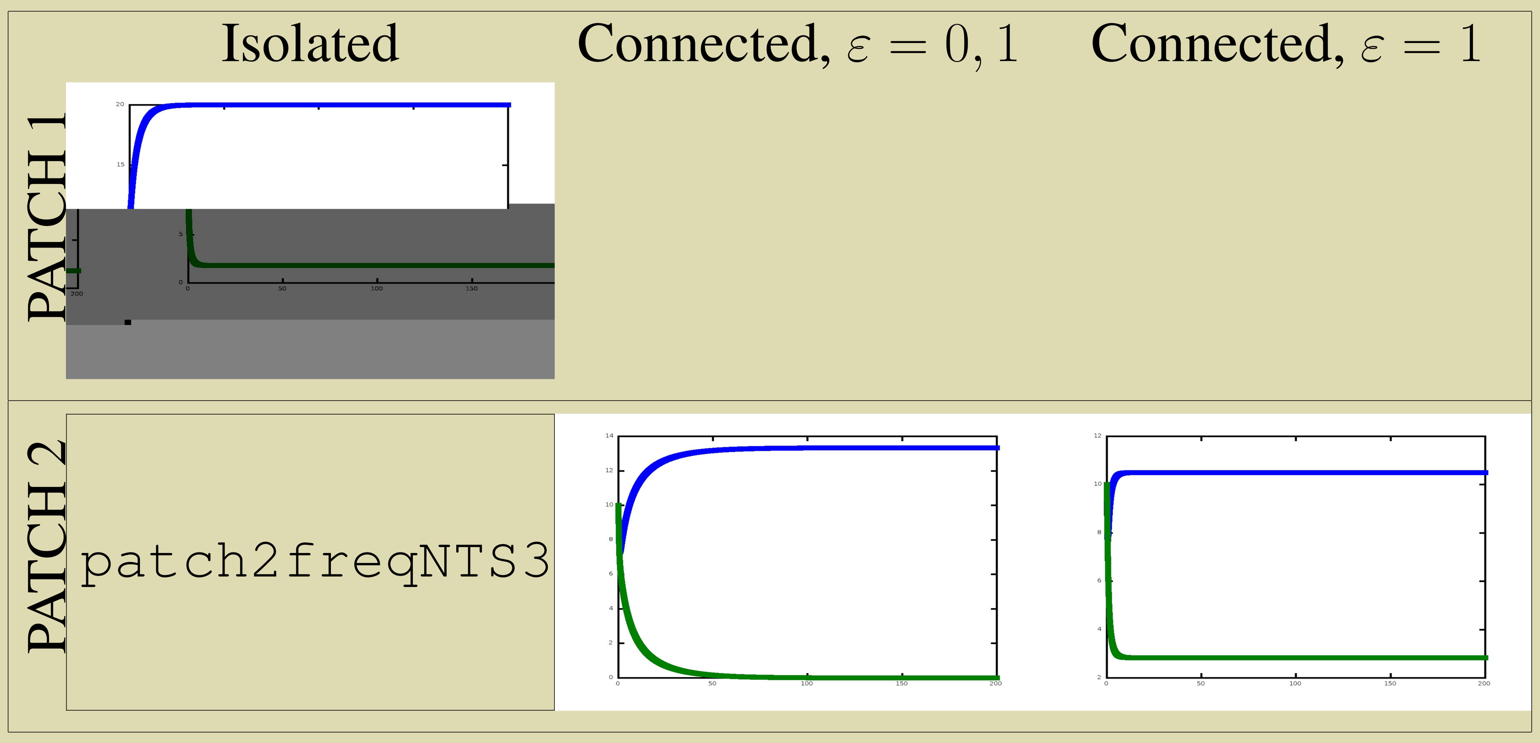
Autonomous case. If parameters in system (3) are constant. Then,

Theorem 1. *Consider system (3) and define the global reproductive number* $\bar{R}_0 := \dfrac{\sum_j \nu_j \beta_j}{\sum_j \nu_j \gamma_j}$. *Then,* $\exists \varepsilon_0 > 0$ *such that* $\forall \varepsilon \in (0, \varepsilon_0)$ *if* $\bar{R}_0 < 1$ *then epidemics is eradicated and if* $\bar{R}_0 > 1$ *epidemics becomes endemic. In this case*

$$\lim_{\tau \to \infty} (S_1(\tau), I_1(\tau) \cdots, S_N(\tau), I_N(\tau)) = (\mu_1 S^*, \nu_1 I^*, \cdots, \mu_N S^*, \nu_N I^*) \qquad (5)$$

where (S^*, I^*) *is the unique positive equilibrium point of system (4).*

Time scales: does it matter? The figure shows the outcome of system (3) ($N = 2$) when we consider (or not) displacements and we distinguish (or not) time scales:



Evolution in time of the number of susceptible (blue) and infected individuals (green). Parameter values $m^S_{12} = 1$; $m^S_{21} = 2$; $m^I_{11} = 1$; $m^I_{21} = 2$; $\beta_1 = 2$; $\beta_2 = 3$; $\gamma_1 = 4$; $\gamma_2 = 2$

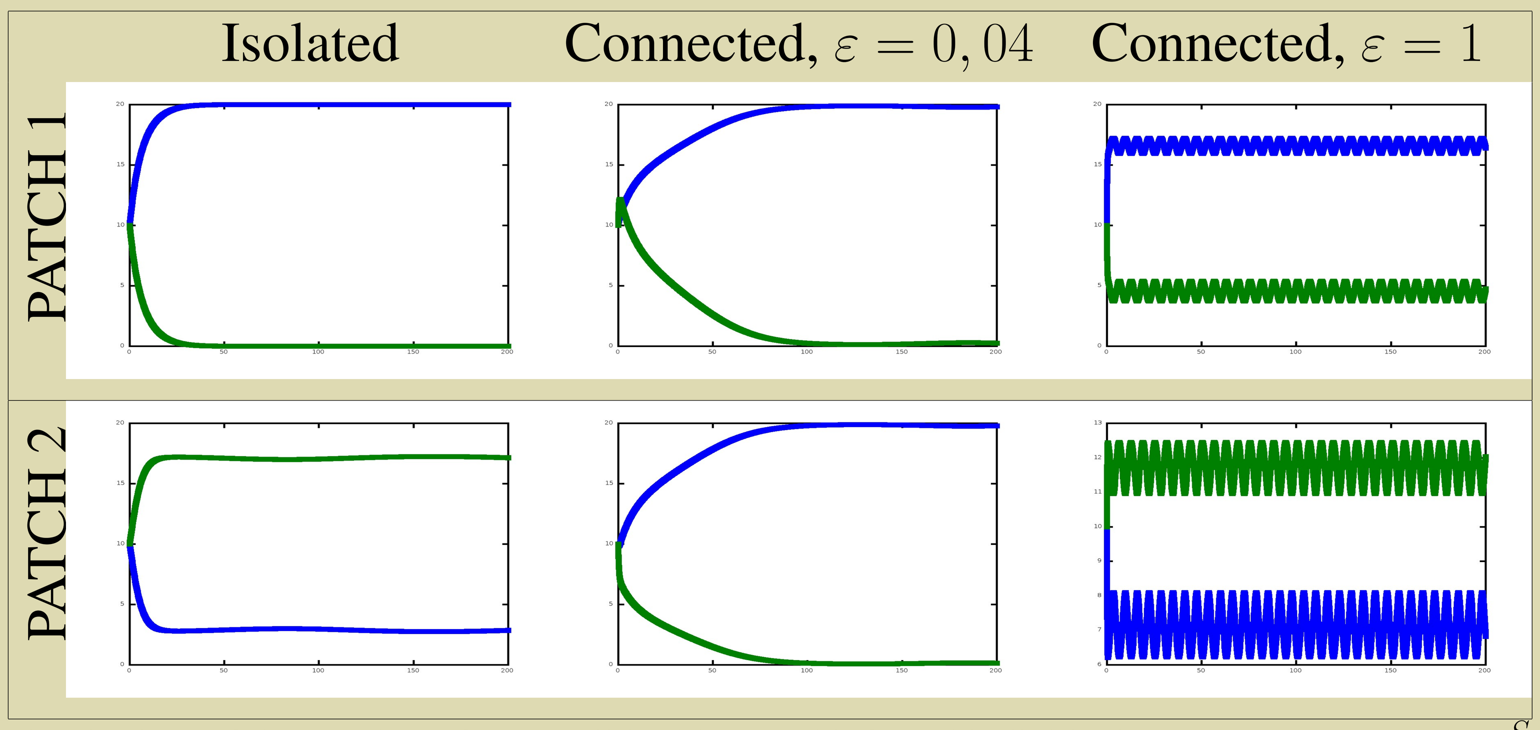
Nonautonomous case. We re-estate theorem 1 following [4]

Theorem 2. *Assume that the epidemic parameters in system (3) are ω-periodic functions of time and define* $\bar{R}_0 := \dfrac{\sum_j \nu_j \int_0^\omega \beta_j(t)dt}{\sum_j \nu_j \int_0^\omega \gamma_j dt}$. *Then, for each* $\xi > 0$ *exist* ε_ξ, $t_\xi > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_\xi)$ *and* $t > t_\xi$

$$\| (S_1(t), I_1(t) \cdots, S_N(t), I_N(t)) - (\mu_1 S^*, \nu_1 I^*, \cdots, \mu_N S^*, \nu_N I^*) \| < \xi,$$

where:
$\bar{R}_0 < 1 \Rightarrow$ *epidemics is eradicated, i.e.* $I^* = 0$, $S^* \equiv$ *total population size.*
$\bar{R}_0 > 1 \Rightarrow (S^*, I^*)$ *is the unique positive periodic solution of system (4).*

Next figure displays the periodic counterpart of the previous figure:



Evolution in time of the number of susceptible (blue) and infected individuals (green). Param. val. $m^S_{12} = 1$; $m^S_{21} = 2$; $m^I_{11} = 1$; $m^I_{21} = 2$; $\beta_1 = 1 + \cos(t)$; $\beta_2 = 7 + \cos(t)$; $\gamma_1 = 5 + 0.1\cos(t)$; $\gamma_2 = 1 + 0.1\cos(t)$

Ongoing work and further results.

- Numerical simulations show that epidemics evolve differently when time scales are (or not) considered. We seek for explicit comparatives between the corresponding reproductive numbers.
- The nonautonomous case admits periodic fast dynamics and asymptotically autonomous terms.

References

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[4] M. Marvá, J.-C. Poggiale, R. Bravo de la Parra. (2012) *Approximate aggregation of a two time scales periodic multi-strain SIS epidemic model: a patchy environment with fast migrations.*Ecological Complexity.**10**, 34–41.

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