Reduction of slow-fast discrete models

Applications to population dynamics

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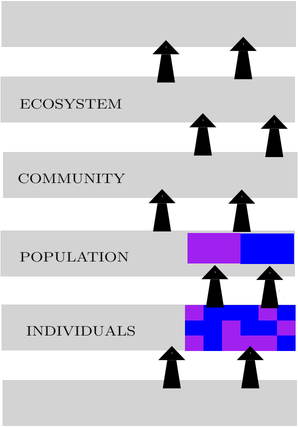
Outline

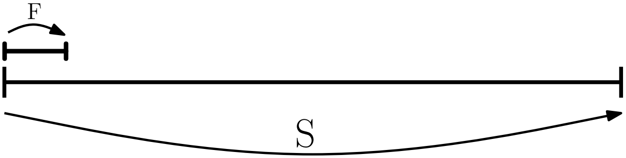
Introduction

Hierarchy theory and structured populations



- Hierarchic organization levels
- Internal strong interactions
- Structured populations
- Time scales and up-scaling

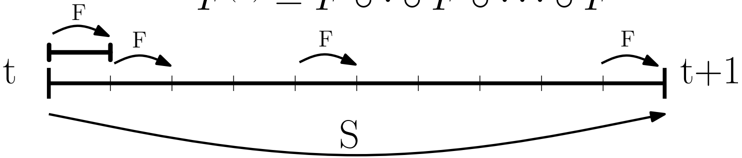
F fast process
S slow process



F fast process

S slow process

$$F^{(k)} = \overbrace{F \circ \cdot \circ F \circ \cdots \circ F}^{k}$$

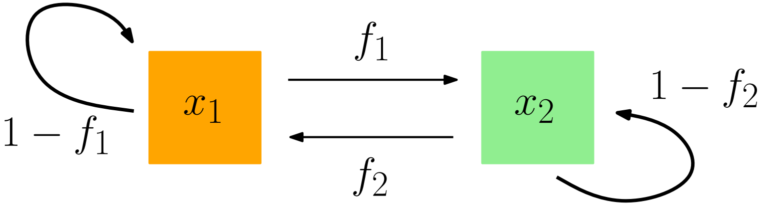


$$X(t+1,k) = S \circ F^{(k)}(X(t,k))$$

A simple example

Population inhabiting two patches

Fast process: individuals can change patch



$$\begin{pmatrix} x_1(\tau+1) \\ x_2(\tau+1) \end{pmatrix} = \begin{pmatrix} 1-f_1 & f_2 \\ f_1 & 1-f_2 \end{pmatrix} \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix}$$

A simple example

Population inhabiting two patches

Slow process: local demography

Malthus	Beverton-Holt
x_1	x_2

$$x_1(t+1) = r_1 x_1(t) \quad x_2(t+1) = \frac{r_2}{1+x_2(t)/M} x_2(t)$$

A simple example: fast dynamics equilibrium

Given the two time scales system

$$\begin{pmatrix} x_1(t+1,k) \\ x_2(t+1,k) \end{pmatrix} = S \circ \left(\begin{pmatrix} 1-f_1 & f_2 \\ f_1 & 1-f_2 \end{pmatrix}^k \begin{pmatrix} x_1(t,k) \\ x_2(t,k) \end{pmatrix} \right)$$

and given that

$$\lim_{k\to\infty} \begin{pmatrix} 1-f_1 & f_2 \\ f_1 & 1-f_2 \end{pmatrix}^k = \begin{pmatrix} \mu_1 & \mu_1 \\ \mu_2 & \mu_2 \end{pmatrix}, \quad \mu_1 = \frac{f_2}{f_1+f_2}, \ \mu_2 = \frac{f_1}{f_1+f_2}$$

we get the *auxiliary system*

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = S \circ \left(\begin{pmatrix} \mu_1 & \mu_1 \\ \mu_2 & \mu_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)$$

A simple example: decomposition and the global variable

Note that $\begin{pmatrix} \mu_1 & \mu_1 \\ \mu_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$

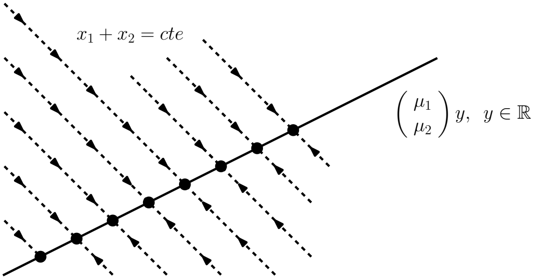
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} S \circ \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)$$

the *global (slow) variable* $\boxed{y = x_1 + x_2}$ yields the *aggregated system*

$$y(t+1) = \left(r_1\mu_1 + \frac{r_2\mu_2}{1 + \mu_2 y(t)/M} \right) y(t)$$

For the fast process

- There exists a center manifold, a global attractor, parametrized by the slow variable
- Slow variables are invariant by the fast dynamics



Results: given y^* a fixed of the reduced system

$$\boxed{X(t+1,k) = S \circ F^{(k)}(X(t+1,k))}$$

$X^* = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} y^*$ is a fixed point of the auxiliary system.

For k large enough, $X_k^* \to X^*$ as $k \to \infty$.

y^* hyperbolic $\Rightarrow X^*, X_k^*$ hyperbolic, same stability as y^*.

The basins of attraction of X_k^* can be analyzed by that of y^*.

Everything holds for periodic orbits.

- L. Sanz, R. Bravo de la Parra, E. Sánchez. *Two time scales non-linear discrete models approximate reduction*. Journal of Difference Equations and Applications, 14 (2008), No. 6, 607–627.

Aggregation of variables

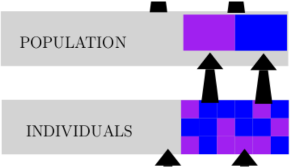
Consider the state variables X (a population)

- divided into q groups (age, size, ...)
- each group divided in A subgroups (individ spread in A zones)

$$X \to (\overbrace{X_1}^{\text{class 1}}, \cdots, \overbrace{X_q}^{\text{class q}}) \to (\overbrace{x_{11}, \cdots, x_{1A}}^{\text{spatial distr class 1}}, \cdots, \overbrace{x_{q1}, \cdots, x_{qA}}^{\text{spatial distr class q}})$$

Fast dynamics internal to each group allows to reduce

$$\boxed{\text{from } \; A \times q \quad \text{eqs system} \; \Rightarrow \quad q \quad \text{eqs system}}$$



$$\begin{pmatrix} \mathcal{F}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \mathcal{F}_2 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \mathcal{F}_q \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_q \end{pmatrix}$$

Approximate aggregation

$$X(t+1,k) = S \circ F^{(k)}(X(t+1,k))$$

$$\Downarrow$$

H1 $$\lim_{k\to\infty} F^{(k)}(X) = \bar{F}(X)$$

$$\Downarrow$$

$$X(t+1,k) = S \circ \bar{F}(X(t+1,k))$$

$$\Downarrow$$

H2 $$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{F = E \circ G} & \mathbb{R}^n \\ & {\scriptstyle G} \searrow \quad \nearrow {\scriptstyle E} & \\ & \mathbb{R}^q & \end{array}$$

$$\Downarrow$$

$$Y(t+1) = G \circ S \circ E(Y(t)) \Leftrightarrow$$ H3 $$\lim_{k\to\infty} DF^{(k)}(X) = D\bar{F}(X)$$

Given F, does exist $\bar{F}$ such that $H1$, $H2$ and $H3$ hold?

- Fast dynamics given by

$$F(X) = \mathrm{diag}(\mathcal{F}_1, \cdots, \mathcal{F}_q)X,$$

where $\mathcal{F}_i$ primitive with dominant eigenvalue 1.

- Fast dynamics given by

$$F(X) = \mathrm{diag}(\mathcal{F}_1(Y), \cdots, \mathcal{F}_q(Y))X,$$

where $\mathcal{F}_i$ primitive with dominant eigenvalue 1, Y global variables.

- M.Marvá, E. Sánchez, R. Bravo de la Parra, L. Sanz. *Reduction of slow-fast discrete models coupling migration and demography*. Journal of theoretical biology. Volume 258. 2009. 371–379.

Given F, does exist $\bar{F}$ such that $H1$, $H2$ and $H3$ hold?

- Matrices $\mathcal{F}_i$ stochastic

$$F(X) = \begin{pmatrix} \mathcal{F}_1 & 0 \\ 0 & \mathcal{F}_2(X_1) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = (\mathcal{F}_1 X_1, \mathcal{F}_2(X_1)X_2),$$

$$\Downarrow$$

$$F^{(k)}(X) = \left(\mathcal{F}_1^k X_1, \prod_{j=0}^{k-1} \mathcal{F}_2(\mathcal{F}_1^{k-j} X_1) X_2 \right)$$

- M. Marvá, R. Bravo de la Parra. *A Juvenile-Adult Model with fast density-dependent migrations* . Submitted.
- T. Nguyen Huu, P. Auger, C. Lett, M. Marvá. *Emergence of global behaviour in a host-parasitoid model with density-dependent dispersal in a chain of patches*. Ecological complexity. Volume 5, Issue 2. 2008. 9–21

Weaker hypothesis

Condition $\lim_{k\to\infty} DF^{(k)}(X) = D\bar{F}(X)$

Can be removed using Liapunov functions

For $F(X) = \mathcal{F}X$, linear primitive with dominant eigenvalue 1

- The decomposition is straightforward.
- Same results for compact attractor instead of equilibrium point.

For a general $F(X)$,

- Same results as for equilibrium point (not uniqueness).
- For any compact attractor: in progress!

Weaker hypothesis

Condition $\lim_{k\to\infty} F^{(k)}(X) = \bar{F}(X)$ can not be removed

Condition $\bar{F} = E \circ G$ can not be removed

This condition is related to a center manifold result:

- Consider $Eq_\Omega(F)$, the set of fixed points of F.
  - The set $Eq_\Omega(F)$ will be the center manifold.
  - $\bar{F}$ projects the space on $Eq_\Omega(F)$.
  - $Eq_\Omega(F)$ must be, *at least*,
    - connected,
    - unbounded,
    - global attractor,
    - parametrized by the global variables.

Still in progress!!

Competitive exclusion principle

Two competing species can coexist only under weak competition.

Possible theoretical outcomes (for Park *et al.*):

- Coexistence.
- Unconditional exclusion of one species.
- One species dies out, depending on initial population sizes.

Observed, experimental, outcomes (flour beetle):

- All the previous ones.
- Coexistence or one species exclusion depending on initial population sizes.

Two patches competing species with fast movements

Notations: $\vec{N} = (N_1^1, N_1^2, N_2^1, N_2^2) = (\vec{N}_1, \vec{N}_2), \qquad n_j = N_j^1 + N_j^2$

Fast dynamics: constant displacement $0 < p_j, q_j < 1$

$$\mathcal{F}_j = \begin{pmatrix} 1-p_j & q_j \\ p_j & 1-q_j \end{pmatrix},$$

Slow dynamics: Leslie-Gower competition at patch j

$$\begin{cases} N_1^j(t+1) = \dfrac{b_1^j}{1 + c_{11}^j N_j^1(t) + c_{12}^j N_2^j(t)} N_1^j(t) \\ N_2^j(t+1) = \dfrac{b_2^j}{1 + c_{21}^j N_1^j(t) + c_{22}^j N_2^j(t)} N_2^j(t) \end{cases} \qquad b_i^j, c_{ij}^k > 0$$

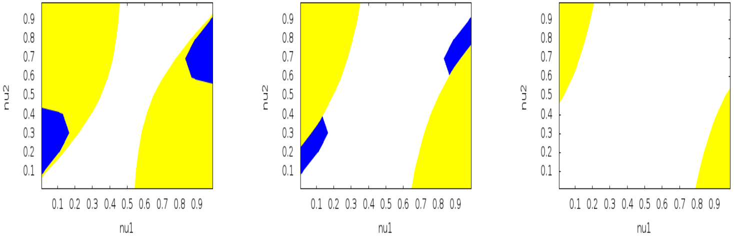
- M. Marvá, R. Bravo de la Parra. *Coexistence and superior competitor exclusion in Leslie-Gower competition model with dispersal*. Submitted.

Two competing species, two patches, fast migrations

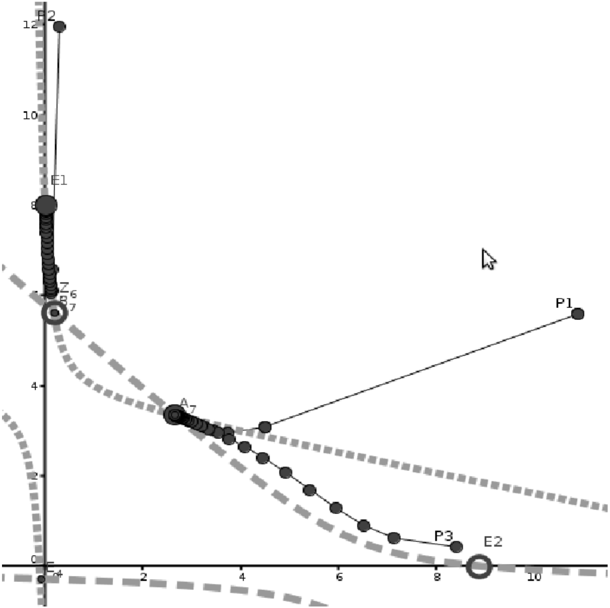
Homogeneous patches

μ_j asymptotic fraction of individuals of species at patch 1

$b = 1.5, \mathbf{c_{21} = 1.1}, c_{12} = 0.9$ $\quad$ $b = 1.5, \mathbf{c_{21} = 1.5}, c_{12} = 0.9$ $\quad$ $b = 1.5, \mathbf{c_{21} = 3}, c_{12} = 0.9$



White	species 1 globally excludes species 2
Yellow	global coexistence
Blue	species 2 globally excludes species 1


P2
12
10
E1
8
Z6
B7
P1
4
A7
2
P3
E2
0
2
4
6
8
10

Questions