A discrete competition eco-epidemiological model

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Basic hipotheses

Empirical evidences of

1. Disease-induced competition outcome reversal
2. Disease-mediated coexistence
3. Disease-boosted extinction

Model that captures these behavior

Many different settings, but **focus on**

- Shared, no-shared, **specialist disease/parasite**
- Explotative, **interference competition**

Two processes which **time units** can be either similar or **quite different**

Epidemic: SIS model

$$(N_{1S}(t+1),\, N_{1I}(t+1)) = F_1(N_{1S}(t),\, N_{1I}(t))$$

$$\begin{aligned} F_1(N_{1S}, N_{1I}) &= (F_{1S}(N_{1S}, N_{1I}), F_{1I}(N_{1S}, N_{1I})) \\ &= \left(N_{1S} - \frac{\beta N_{1S} N_{1I}}{N_{1S}+N_{1I}} + \gamma N_{1I},\ N_{1I} + \frac{\beta N_{1S} N_{1I}}{N_{1S}+N_{1I}} - \gamma N_{1I}\right) \end{aligned}$$

1. Assume $\gamma \leq 1$, $\beta < \left(1+\sqrt{\gamma}\right)^2$, $\gamma < \beta < 2+\gamma$
2. Constant total population size
3. Basic reproduction number: $R_0 = \beta/\gamma$

$$\lim_{t\to\infty} (N_{1S}(t),\, N_{1I}(t)) = \begin{cases} N^*_{1df} = (N_1, 0) & \text{if } R_0 \leq 1 \\ N^*_{1e} = (\nu, (1-\nu))\, N_1 & \text{if } R_0 > 1 \end{cases}, \; GAS,\; \nu = \frac{1}{R_0}$$

L J Allen. Some discrete-time SI, SIR, and SIS epidemic models. Math Biosc, 124(1):83-105, 1994

Competition: Leslie-Gower model

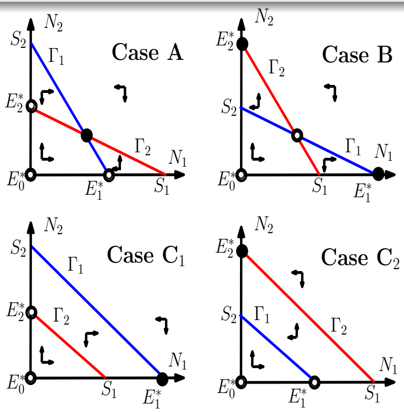
$$N_1(t+1) = \frac{b_1}{1+c_{11}N_1(t)+c_{12}N_2(t)} N_1(t)$$

$$N_2(t+1) = \frac{b_2}{1+c_{21}N_1(t)+c_{22}N_2(t)} N_2(t)$$

- $b_i \leq 1$ then $N_i(t) \to 0$
- $b_i > 1$ species can survive alone

- Forward bounded solutions
- **Solutions are eventually componentwise monotone**

- Γ_i points whose i-coordinate is held fixed by the map



J.M. Cushing et al, 2004. Some Discrete Competition Models and the Competitive Exclusion Principle JDEA, **10**(13-15): 1139-1151

HL Smith 1998. Planar competitive and cooperative difference equations. JDEA, 3(5-6):335-357

Competition: Leslie-Gower model with infected individuals

Especialist disease/parasite

$$N_{1S}(t+1) = \frac{b_{1S}N_{1S}(t)}{1 + c_{SS}N_{1S}(t) + c_{SI}N_{1I}(t) + c_{S2}N_2(t)}$$

$$N_{1I}(t+1) = \frac{b_{1I}N_{1I}(t)}{1 + c_{IS}N_{1S}(t) + c_{II}N_{1I}(t) + c_{I2}N_2(t)}$$

$$N_2(t+1) = \frac{b_2N_2(t)}{1 + c_{2S}N_{1S}(t) + c_{2I}N_{1I}(t) + c_{22}N_2(t)}$$

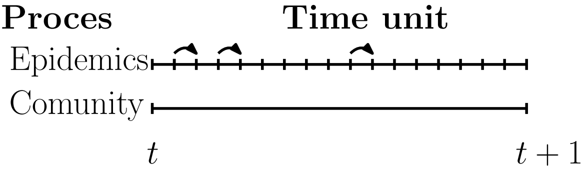
Competition epidemiological model with two time scales

Proces **Time unit**

Epidemics

Comunity

Competition epidemiological model with two time scales



Notation

S: competition, demography
F: epidemics

Two time scales model or slow-fast system

$$N(t+1) = S \circ \overbrace{F \circ F \circ \cdots \circ F}^{k \ times}(N(t)) = S \circ F^{(k)}(N(t))$$

Competition epidemiological model with two time scales

$$N(t+1) = S(F^{(k)}(N(t)))$$

$$N_{1S}(t) \equiv F_{1S}^{(k)}(N_{1S}(t), N_{1I}(t)) \qquad N_{1I}(t) \equiv F_{1I}^{(k)}(N_{1S}(t), N_{1I}(t))$$

$$N_{1S}(t+1) = \frac{b_{1S}F_S^{(k)}(N_{1S}(t), N_{1I}(t))}{1 + c_{SS}F_S^{(k)}(N_{1S}(t), N_{1I}(t)) + c_{SI}F_I^{(k)}(N_{1S}(t), N_{1I}(t)) + c_{S2}N_2(t)}$$

$$N_{1I}(t+1) = \frac{b_{1I}F_I^{(k)}(N_{1S}(t), N_{1I}(t))}{1 + c_{IS}F_S^{(k)}(N_{1S}(t), N_{1I}(t)) + c_{II}F_I^{(k)}(N_{1S}(t), N_{1I}(t)) + c_{I2}N_2(t)}$$

$$N_2(t+1) = \frac{b_2N_2(t)}{1 + c_{2S}F_S^{(k)}(N_{1S}(t), N_{1I}(t)) + c_{2I}F_I^{(k)}(N_{1S}(t), N_{1I}(t)) + c_{22}N_2(t)}$$

Slow-fast discrete models

Given the slow-fast system

$$N(t+1) = S(F^{(k)}(N(t)))$$

assume that the fast dynamics is instantaneus

$$\begin{aligned}\lim_{k\to\infty} F^{(k)}(N) &= \lim_{k\to\infty} \left(F_{1S}^{(k)}(N_{1S}, N_{1I}), F_{1I}^{(k)}(N_{1S}, N_{1I}), N_2\right) \\ &= (\nu N_1,\ (1-\nu)N_1,\ N_2) \\ &= \bar{F}(N)\end{aligned}$$

allows to approximate the original system by the *auxiliary system*

$$N(t+1) = S(\bar{F}(N(t)))$$

Slow-fast discrete models: dimension reduction

Given that

$$\bar{F}(N) = \begin{pmatrix} \nu & 0 \\ 1-\nu & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} N_{1S} \\ N_{1I} \\ N_2 \end{pmatrix} = E \circ G(N)$$

$N(t+1) = S \circ E \circ G(N(t))$ $\Rightarrow$ slow variables $G(N) := (N_1, N_2)$ $\Rightarrow$

$G(N(t+1)) = G \circ S \circ E \circ G(N(t))$

$$N_1(t+1) = \frac{\nu b_{1S} N_1(t)}{1 + (\nu c_{SS} + (1-\nu) c_{SI}) N_1(t) + c_{S2} N_2(t)} + \frac{(1-\nu) b_{1I} N_1(t)}{1 + (\nu c_{IS} + (1-\nu) c_{II}) N_1(t) + c_{I2} N_2(t)}$$

$$N_2(t+1) = \frac{b_2 N_2(t)}{1 + (\nu c_{2S} + (1-\nu) c_{2I}) N_1(t) + c_{22} N_2(t)}$$

Approximate aggregation $N(t+1) = S \circ F^{(k)}(N(t))$

Assume that the limits are uniform on compact sets

$$\lim_{k\to\infty} F^{(k)} = \bar{F} \qquad \text{and} \qquad \lim_{k\to\infty} DF^{(k)} = D\bar{F}$$

Let $n^* = (n_1^*, n_2^*) \in \mathbb{R}^2$ be a hyperbolic fixed point of the reduced system.

1. For all $k \geq k_0$ the slow-fast system has an unique fixed point N_k^* which is hyperbolic and
$$\lim_{k\to\infty} N_k^* = (\nu n_1^*,\ (1-\nu)n_1^*,\ n_2^*)$$
2. n^* is A.S. (U.) $\Leftrightarrow$ N_k^* is A.S. (U.) for all $k \geq k_1$.
3. The basin of attraction of N_k^* can be described from that of n^*.

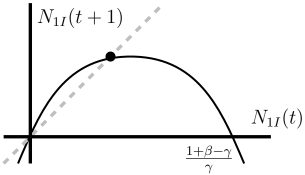
L. Sanz, R. Bravo de la Parra, E. Sánchez (2008). *Two time scales non-linear discrete models approximate reduction*. JDEA, 14, No. 6, 607–627.

Approximate aggregation: uniform convergence

The map

$$\phi(x) = x\,(1 + \beta(1 - x) - \gamma)$$

updates the fraction of infected individuals



So that

$$F_1^{(k)}(N_{1S}, N_{1I}) = \left((1 - \phi^{(k)}(N_{1I}/N_1))N, \phi^{(k)}(N_{1I}/N_1)N_1\right),$$

besides

$$DF_1^{(k)}(N_{1S}, N_{1I}) = \begin{pmatrix} 1 - \phi^{(k)}(N_{1I}/N_1) & 1 - \phi^{(k)}(N_{1I}/N_1) \\ \phi^{(k)}(N_{1I}/N_1) & \phi^{(k)}(N_{1I}/N_1) \end{pmatrix}$$

$$+ \quad \left(\phi^{(k)}(N_{1I}/N_1)\right)' \begin{pmatrix} N_{1I}/N_1 & N_{1I}/N_1 - 1 \\ -N_{1I}/N_1 & 1 - N_{1I}/N_1 \end{pmatrix}$$

Reduced Discrete SIS-competition model

$$N_1(t+1) = \frac{\nu b_{1S} N_1(t)}{1+(\nu c_{SS}+(1-\nu)c_{SI})N_1(t)+c_{S2}N_2(t)} + \frac{(1-\nu)b_{1I}N_1(t)}{1+(\nu c_{IS}+(1-\nu)c_{II})N_1(t)+c_{I2}N_2(t)}$$

$$N_2(t+1) = \frac{b_2 N_2(t)}{1+(\nu c_{2S}+(1-\nu)c_{2I})N_1(t)+c_{22}N_2(t)}$$

All solutions in $\mathbb{R}^2_+$ are

1. Forward bounded
2. Eventually componentwise monotone

Besides

1. If $\nu b_{1S} + (1-\nu)b_{1I} \leq 1$ then $N_1(t) \to 0$
2. If $b_2 \leq 1$ then $N_2(t) \to 0$
3. Otherwise, species can survive alone

R. Bravo de la Parra, M. Marvá, E. Sánchez, L. Sanz Discrete Models of Disease and Competition (*Submitted*)

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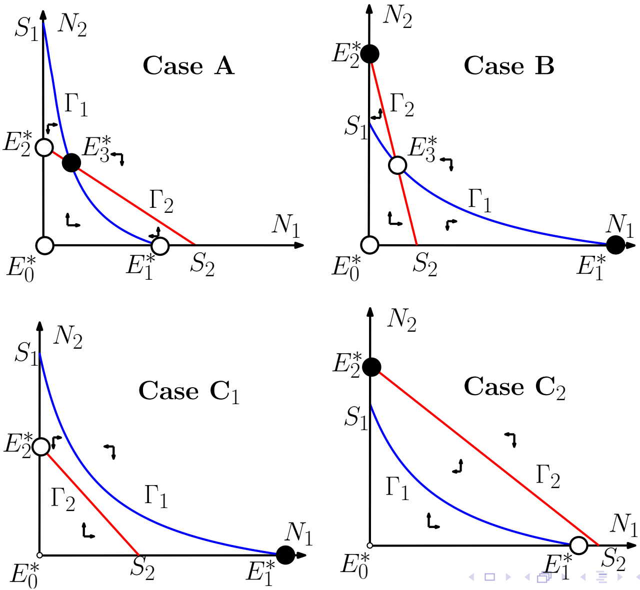
Reduced Discrete SIS-competition model



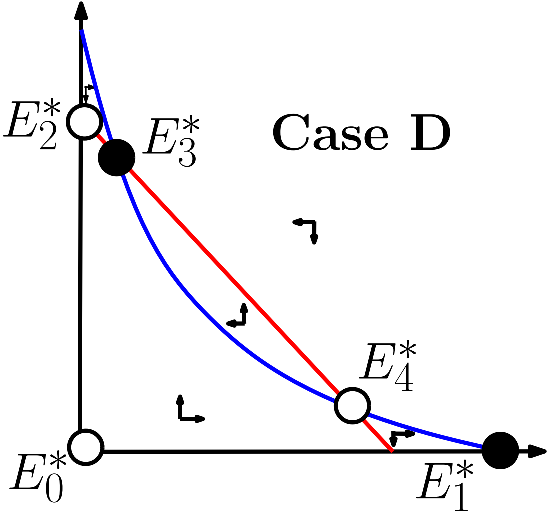


Figure: **Case C$_2$**. Bi-stability with interior equilibrium point

Disease mediated competition (affects growth capabilities)

$$c_{11} := c_{SS} = c_{SI} = c_{IS} = c_{II}, \quad c_{12} := c_{S2} = c_{I2}, \quad c_{21} = c_{2S} = c_{2I}.$$

Case A (coexistence).

$$b_{1S} = b_1 \qquad b_{1I} = \alpha b_1 \qquad \alpha > 0 \text{ effect of the disease}$$

Ratio of each species population size at equilibrium with/without disease

$$\frac{N_{1e}^*}{N_{1df}^*} = \frac{c_{22}\left(b_1\left[\nu + (1-\nu)\,\alpha\right] - 1\right) - c_{12}\left(b_2 - 1\right)}{c_{22}\left(b_1 - 1\right) - c_{12}\left(b_2 - 1\right)}$$

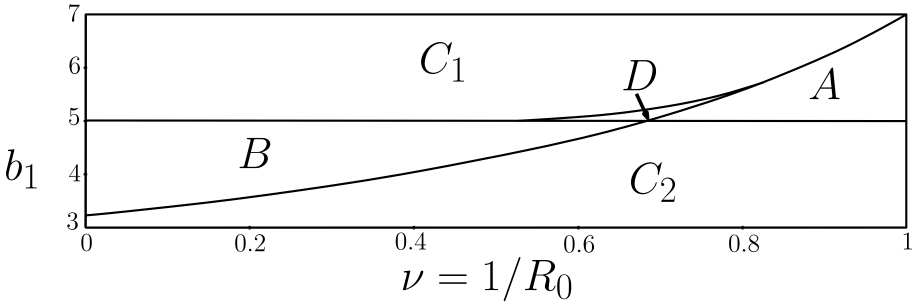
$$\frac{N_{2e}^*}{N_{2df}^*} = \frac{c_{11}\left(b_2 - 1\right) - c_{21}\left(b_1\left[\nu + (1-\nu)\,\alpha\right] - 1\right)}{c_{11}\left(b_2 - 1\right) - c_{21}\left(b_1 - 1\right)}$$

Disease modified competition (affects competitive abilities)

$$b_1 := b_{1S} = b_{1I} > 1,$$

$$c_{SS} = 3 > c_{SI} = 2.8, \qquad c_{2S} = 2 > c_{2I} = 1.8,$$

$$b_2 = 5, \qquad c_{S2} = c_{IS} = c_{II} = c_{I2} = c_{22} = 1, \qquad \nu \in (0, 1]$$



Increasing R_0 improves the species 1 competition outcome.

THANK YOU!

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