Discrete models of disease and competition

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Universidad de Alcalá

- Opened up in 1499.
- Moved to Madrid city in 1836 (UCM).
- Back to Alcalá in 1970.

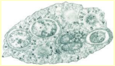
Empirical evidences of diseases/parasites-population interactions

Anolis gingivinus

Plasmodium azurophilum

Anolis Watts

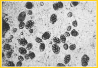
Bemisia tabaci



Wolbachia

Tribolium confusum

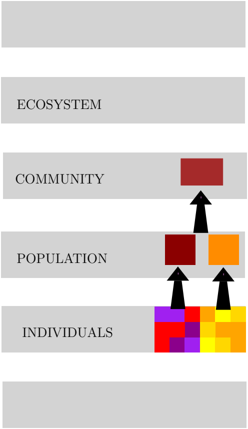
Adelina tribolii



Tribolium castaneum

- Global change.
- Diseases as populations control.

Organization levels and time scales



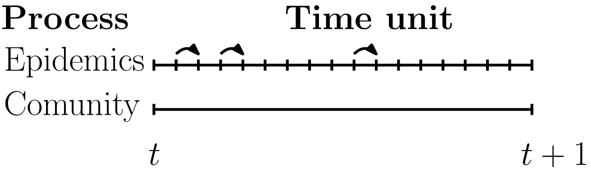
Hierarchical organization levels

- Different levels, different time scales
- Subgroups with strong interactions
- Heterogeneity may define subgrups:
  - Epidemiological state
  - Individual traits
  - Spatial distribution
  - Social status

Objectives:

1. Building up models that capture previous behavior.
2. Models linking levels.

Process	Time unit
Epidemics	
Comunity	



Notation

$\mathcal{S}$: competition, demography
F: epidemics

Two time scales model or slow-fast system

$$N_{t+1} = \mathcal{S} \circ \overbrace{F \circ F \circ \cdots \circ F}^{k\ times}(N_t) = \mathcal{S} \circ F^{(k)}(N_t)$$

Dimension reduction process

Given the prototype

$$X_{t+1} = \mathcal{S} \circ F^{(k)}(X_t) \tag{1}$$

(H1) $\forall X \in \Omega_N$ there exists $\bar{F}(X) := lim_{k\to\infty} F^{(k)}(X)$

$$X_{t+1} := \mathcal{S} \circ \bar{F}(X_t) \qquad \text{auxiliary system} \tag{2}$$

(H2) If there exist $\Omega_q \subset \mathbb{R}^q$, where $q < N$ and $\Omega_N \xrightarrow{G} \Omega_q \xrightarrow{E} \Omega_N$ such that $\bar{F} = E \circ G$

$$Y_{t+1} = G \circ \mathcal{S} \circ E\,(Y_t) \ \ \text{reduced system,} \qquad \text{slow variables} \quad \boxed{Y = G(X)} \tag{3}$$

Theorem Assume H1, H2. Let $Y^* \in \mathbb{R}^q$ be a hyperbolic equilibrium of (3), then

1. $X^* = E(Y^*)$ hyperbolic equilibrium of (2).

Under suitable convergence $F^{(k)} \to \bar{F}$, for k large enough:

1. There exist $X_k^* \to X^*$ equil of (1).
2. The stability of Y^*, X^*, X_k^* is the same.
3. The basins of attraction of X^*, X_k^* cab described by that of Y^*.

Fast process: SIS epidemics

$$\begin{pmatrix} S \\ I \end{pmatrix} \xrightarrow{F} \begin{pmatrix} S - \frac{\beta IS}{S+I} + \gamma I \\ I + \frac{\beta IS}{S+I} - \gamma I \end{pmatrix} \qquad \lim_{k\to\infty} F^{(k)}(S,\, I) = \begin{cases} (N,0), & if\, R_0 \leq 1 \\ (\nu,(1-\nu))\,N, & if\, R_0 > 1 \end{cases}$$

$$\nu = \frac{1}{R_0}$$

1. Constant total population size: $N = S(t) + I(t)$
2. Where $R_0 = \beta/\gamma$ is the *basic reproduction number*.

Slow process: Beverton-Holt population dynamics

$$N(t+1) = \frac{b}{1+cN(t)}\,N(t), \Rightarrow \lim_{t\to\infty} N(t) = \begin{cases} 0 & if \;\; b \leq 1 \\ N^* = \frac{b-1}{c} & if \;\; b > 1 \end{cases}$$

Single species two time scales epidemic model: $N(t+1) = \mathcal{S}\left(F^{(k)}N(t)\right)$

Complete system: we distinguish S and I

$$\begin{cases} S(t+1) = \dfrac{b_S}{1 + c_{SS}S(t) + c_{SI}I(t)}\, S(t) \\[2ex] I(t+1) = \dfrac{b_I}{1 + c_{IS}S(t) + c_{II}I(t)}\, I(t) \end{cases}$$

But **also SIS is faster than population dynamics** $\quad S(t) \equiv F_S{}^{(k)}\,(S(t), I(t)) \qquad I(t) \equiv F_I^{(k)}(S(t), I(t))$

Auxiliary system: $k \to \infty$

$$\begin{cases} S(t+1) = \dfrac{b_S \nu N(t)}{1 + c_{SS}\nu N(t) + c_{SI}(1-\nu)N(t)} \\[2ex] I(t+1) = \dfrac{b_I(1-\nu)N(t)}{1 + c_{IS}\nu N(t) + c_{II}(1-\nu)N(t)} \end{cases}$$

Reduced system: $N = S + I \qquad N(t+1) = \dfrac{b_1 N(t)}{1 + c_1 N(t)} + \dfrac{b_2 N(t)}{1 + c_2 N(t)}$

Results 1: persistence or extinction

If $\nu b_S + (1-\nu) b_I \leq 1$ then

$$\lim_{t\to\infty} N(t) = 0 \qquad \Leftrightarrow \qquad \lim_{t\to\infty} (S(t),\, I(t)) = (0,\, 0)$$

For instance, if

- $b_S,\ b_I < 1$
- $b_S > 1$ and $b_I < \frac{1-\nu b_S}{1-\nu}$

If $\nu b_S + (1-\nu) b_I > 1$ then

$$\lim_{t\to\infty} N(t) = N^* = \frac{\left((b_2-1)c_1 + (b_1-1)c_2 + \sqrt{((b_2-1)c_1 + (b_1-1)c_2)^2 + 4(b_1+b_2-1)c_1c_2}\right)}{2c_1c_2}$$

$$\Leftrightarrow \qquad \lim_{t\to\infty} (S(t),\, I(t)) \approx (\nu N^*,\, (1-\nu)\, N^*)$$

For instance, if

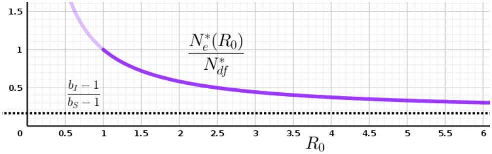
- $b_S,\ b_I > 1$
- $b_S < 1$ and $b_I > \frac{1-\nu b_S}{1-\nu}$

Results 2: disease mediated competition (affects growth)

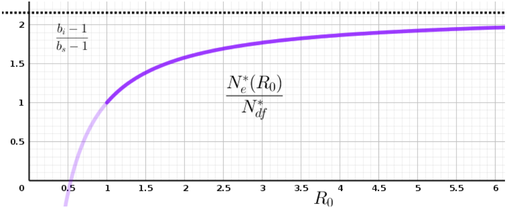
That is, $c_{SS} = c_{II} = c_{SI} = c_{IS}$, then

$$\frac{N_e^*}{N_{df}^*} = \frac{b_S - b_I}{b_S - 1}\frac{1}{R_0} + \frac{b_I - 1}{b_S - 1}$$

Disease reduced fecundity $b_I < b_S$

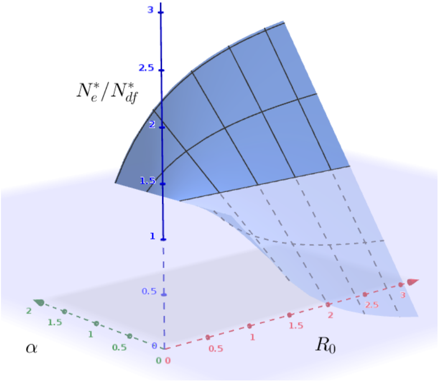


Disease enhanced fecundity $b_I > b_S$



That is, $c_{SS} = c_{II} = c_{SI} = c_{IS}$ and $b_I = \alpha b_S$, then

$$\frac{N_e^*}{N_{df}^*} = \frac{b_S - \alpha b_S}{b_S - 1} \frac{1}{R_0} + \frac{\alpha b_S - 1}{b_S - 1}$$



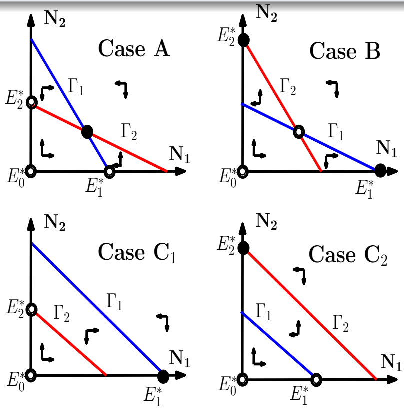
$$N_1(t+1) = \frac{b_1}{1+c_{11}N_1(t)+c_{12}N_2(t)}N_1(t)$$

$$N_2(t+1) = \frac{b_2}{1+c_{21}N_1(t)+c_{22}N_2(t)}N_2(t)$$

- $b_i \le 1$ then $N_i(t) \to 0$
- $b_i > 1$ species i can survive alone

- Forward bounded solutions
- **Solutions are eventually componentwise monotone**

- Γ_i points whose i-coordinate is held fixed by the map



J.M. Cushing et al, 2004. Some Discrete Competition Models and the Competitive Exclusion Principle JDEA, **10**(13-15): 1139-1151
HL Smith 1998. Planar competitive and cooperative difference equations. JDEA, 3(5-6):335-357

Two time scales competition with specialist parasite

$$N(t+1) = \mathcal{S}(F^{(k)}(N(t)))$$

Species 1: S and I

$$S(t) \equiv F_S^{(k)}(S(t), I(t)) \qquad I(t) \equiv F_I^{(k)}(S(t), I(t))$$

$$S_1(t+1) = \frac{b_S F_S^{(k)}(S(t), I(t))}{1 + c_{SS} F_S^{(k)}(S(t), I(t)) + c_{SI} F_I^{(k)}(S(t), I(t)) + c_{S2} N_2(t)}$$

$$I_1(t+1) = \frac{b_I F_I^{(k)}(S(t), I(t))}{1 + c_{IS} F_S^{(k)}(S(t), I(t)) + c_{II} F_I^{(k)}(S(t), I(t)) + c_{I2} N_2(t)}$$

$$N_2(t+1) = \frac{b_2 N_2(t)}{1 + c_{2S} F_S^{(k)}(S(t), I(t)) + c_{2I} F_I^{(k)}(S(t), I(t)) + c_{22} N_2(t)}$$

Reduced Discrete SIS-competition model

$$N_1(t+1) = \frac{\nu b_S N_1(t)}{1+(\nu c_{SS}+(1-\nu)c_{SI})\,N_1(t)+c_{S2}N_2(t)} + \frac{(1-\nu)b_I N_1(t)}{1+(\nu c_{IS}+(1-\nu)c_{II})\,N_1(t)+c_{I2}N_2(t)}$$

$$N_2(t+1) = \frac{b_2 N_2(t)}{1+(\nu c_{2S}+(1-\nu)c_{2I})\,N_1(t)+c_{22}N_2(t)}$$

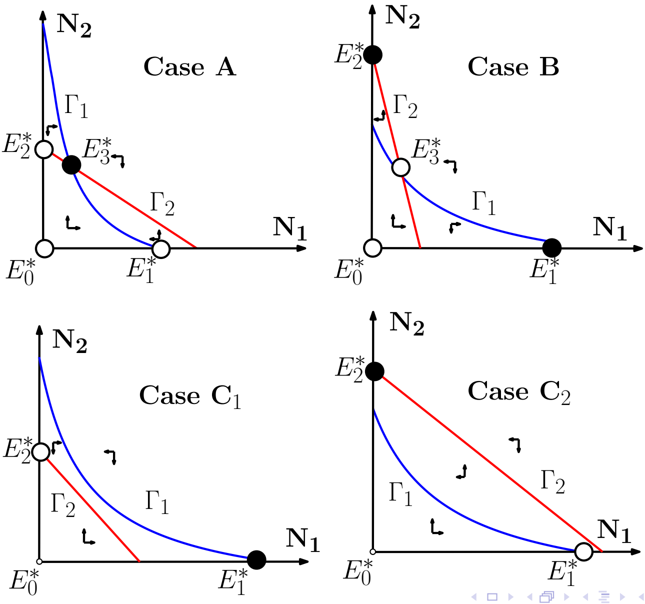
All solutions in $\mathbb{R}^2_+$ are

1. Forward bounded
2. Eventually componentwise monotone

Besides

1. If $\nu b_S+(1-\nu)b_I\leq 1$ then $N_1(t)\to 0$
2. If $b_2\leq 1$ then $N_2(t)\to 0$
3. Otherwise, species can survive alone

R. Bravo de la Parra, M. Marvá, E. Sánchez, L. Sanz 2017. Discrete Models of Disease and Competition (*Discrete Dynamics in Nature and Society, Article ID 5310837*)
HL Smith 1998. Planar competitive and cooperative difference equations. JDEA, 3(5-6):335-357


N₂
Case A
Γ_1
E_2^*
E_3^*
Γ_2
N₁
E_0^*
E_1^*
N₂
E_2^*
Case B
Γ_2
E_3^*
Γ_1
N₁
E_0^*
E_1^*
N₂
Case C₁
E_2^*
Γ_2
Γ_1
N₁
E_0^*
E_1^*
N₂
E_2^*
Case C₂
Γ_1
Γ_2
N₁
E_0^*
E_1^*

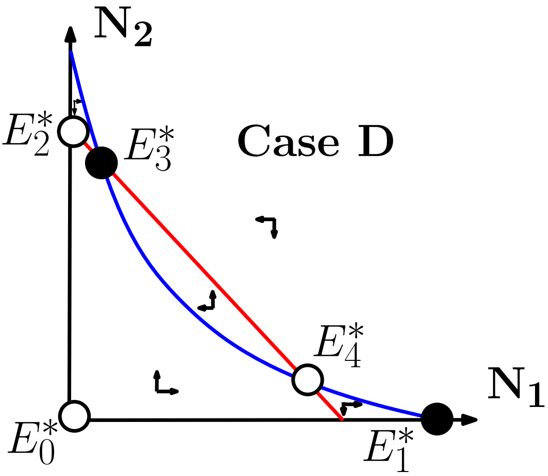


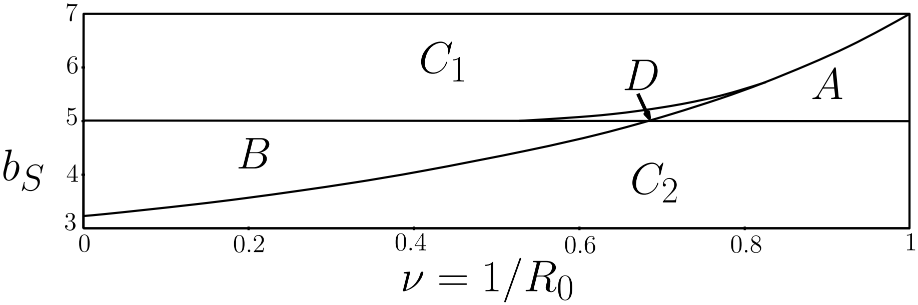
Figure: **Case D**. Bi-stability with interior equilibrium point

Disease modified competition (affects competitive abilities)

$$b_S = b_I > 1,$$

$$c_{SS} = 3 > c_{SI} = 2.8, \qquad c_{2S} = 2 > c_{2I} = 1.8,$$

$$b_2 = 5, \qquad c_{S2} = c_{IS} = c_{II} = c_{I2} = c_{22} = 1, \qquad \nu \in (0, 1]$$



Increasing R_0 improves the species 1 competition outcome.

Disease mediated competition (affects growth)

$$c_{11} := c_{SS} = c_{SI} = c_{IS} = c_{II}, \quad c_{12} := c_{S2} = c_{I2}, \qquad c_{21} := c_{2S} = c_{2I}.$$

Case A (coexistence). Assume

$$b_I = \alpha b_S \qquad \alpha > 0 \text{ effect of the disease}$$

Ratio of each species population size at equilibrium with/without disease in species 1:

$$\frac{N_{1e}^*}{N_{1df}^*} = \frac{c_{22}\left(b_S\left[\nu + (1-\nu)\,\alpha\right] - 1\right) - c_{12}\left(b_2 - 1\right)}{c_{22}\left(b_S - 1\right) - c_{12}\left(b_2 - 1\right)}$$

$$\frac{N_{2e}^*}{N_{2df}^*} = \frac{c_{11}\left(b_2 - 1\right) - c_{21}\left(b_S\left[\nu + (1-\nu)\,\alpha\right] - 1\right)}{c_{11}\left(b_2 - 1\right) - c_{21}\left(b_S - 1\right)}$$

THANK YOU!

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