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SUPERSOLUTIONS TO DEGENERATED LOGISTIC EQUATION TYPE

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Abstract

In this paper, we provide a method for building up a strictly positive supersolution for the steady state of a degenerated logistic equation type, i.e., when the weight function vanishes on the boundary of the domain. This degenerated system is related in obtaining the so-called large solutions. Previously, this problem was handled as the limit case of non degenerated approaching problems. Our method can be adapted straightforwardly to degenerated boundary value problems.

1. Introduction

In this paper, we show how to build up a positive strict supersolution to the boundary value problem:

$$\begin{cases} -\Delta u = \lambda m(x)u - a(x)f(x, u), & \text{in } \Omega, \\ u = g(x), & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $a \in C^{\mu}(\overline{\Omega}; [0, +\infty))$, $m \in C^{\mu}(\overline{\Omega}; \mathbb{R})$, $\mu \in (0, 1)$, $a \equiv 0$ on $\partial\Omega$, $g \in C^{1+\mu}(\partial\Omega; [0, +\infty))$ is such that $g \geq 0$, $g \not\equiv 0$, $f \in C^{\mu, 1+\mu}(\overline{\Omega} \times$

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$[0, +\infty); \mathbb{R})$, $f(x, 0) = 0$, $f(x, u) > 0$, $\partial_u f(x, u) > 0$ and $\lim_{k \nearrow +\infty} \dfrac{f(x, ku)}{k} = +\infty$ for each $u > 0$ uniformly in $x \in \Omega$. The set $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega$ of class C^2.

Problem (1) $(\text{with } m \equiv 1)$ stands for the steady states of the generalized logistic growth law [13]. Typically, u stands for the distribution in Ω of individuals of certain species, λ stands for the net growth rate of u modulated by $m(x)$, $a(x)$ simulates demography pressure and, along with $m(x)$, environmental heterogeneity. When $a \equiv 0$ somewhere, model (1) $(\text{with } m \equiv 1)$ becomes a Malthusian model in the region $\Omega_0 := \{x \in \Omega; a(x) = 0\}$, known as *refuge*. We deal with the case $a \equiv 0$ on $\partial\Omega$, but it could be adapted to apply in the general case when $\Omega_0 \neq \emptyset$ and $\Omega_0 \subset \Omega$.

In addition, solving problem (1) is an intermediate step to prove the existence of large solutions related to problem (1), i.e., a function $u \in C^{2,\nu}(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u = \lambda m(x)u - a(x) f(x, u)u & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} u(x) = \infty, \end{cases} \tag{2}$$

see [11]. Large solutions come across from studies concerning combustion due to Keller [9] and Osserman [14]. In fact, there are available uniqueness results for problem (2) ([3], [5], [12] and references therein). As long as we know, problem (1) has been handled as the limit case of non degenerated approaching problems.

If we restrict ourselves in problem (1) to $a(x) \geq \gamma > 0$ on $\overline{\Omega}$, it is easy to show that there exists $\varepsilon > 0$ small enough such that $\varepsilon\varphi$ is a positive subsolution to problem (1), being φ the principal eigenfunction of $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions. In this case a positive constant K such that

$$K > \max\left\{\max_{x\in\partial\Omega}\{g\}, \frac{\max_{x\in\overline{\Omega}}\{\lambda m(x)\}}{\min_{x\in\overline{\Omega}}\{a(x)\}}\right\}, \tag{3}$$

provides us with a supersolution to problem (1). Now, enlarging K if necessary, we get an ordered pair $(\varepsilon\varphi, K)$ of sub-supersolution. Thanks to a Theorem by Amann [1], there exists a solution to problem (1) between $\varepsilon\varphi$ and K. Unfortunately, as soon as we let $a = 0$ somewhere in $\overline{\Omega}$, we lose condition (3), so it is needed something different from a constant to get a supersolution to problem (1). Different strategies have been used to avoid this problem: some authors approximate Ω by $\Omega_n := \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > 1/n\}$ (for each n $a(x)$ is uniformly bounded away from 0 in Ω_n, that is $a \geq \gamma > 0$, see [4] and [11]), and others approach $a(x)$ by means of $\frac{1}{n} + a(x)$, essentially in order to avoid condition $a \equiv 0$ on $\partial\Omega$ and to generate a sequence of solutions of approximate problems converging to a solution to problem (1), see [2] and [8].

2. Results

The result we present here allows us to obtain via sub and supersolution the existence of a solution to (1). Actually, the construction of the supersolution in Ω follows from an extension of the Faber [6] and Krahn [7] inequality due to López-Gómez [10] which provides us with a lower estimate for the main eigenvalue to $-\Delta$ on the domain D:

$$\sigma_1[-\Delta, D] \geq \frac{\sigma_1[-\Delta, B_1(0)]\cdot |B_1(0)|^{2/N}}{|D|^{2/N}}, \tag{4}$$

where $B_1(0) := \{x \in \mathbb{R}^N; |x| \leq 1\}$, $\sigma_1[-\Delta, B_1(0)]$ is the principal eigenvalue of $-\Delta$ on $B_1(0)$ and $|D|$ stands for the Lebesgue's measure of D.

Theorem 2.1. *Consider the boundary value problem* (1). *Then, for each* $\lambda \in \mathbb{R}$, *there exists an ordered pair* $(\underline{u}, \overline{u})$ *consisting of a subsolution and a positive strict supersolution*, *where* $\underline{u}(x) < \overline{u}(x)\ \forall x \in \Omega$.

Proof. Let us define

$$O_\varepsilon := \{x \in \mathbb{R}^N;\ \mathrm{dist}(x, \partial\Omega) < \varepsilon\}, \quad \sigma_\varepsilon := \sigma[-\Delta, O_\varepsilon],$$

where, keeping in mind (4), for a fixed $\lambda \in \mathbb{R}$ we can choose $\varepsilon > 0$ such that

$$\max_{x \in \overline{\Omega}} \{\lambda m(x)\} < \sigma_\varepsilon.$$

Let φ_o^ε be the principal eigenfunction of $-\Delta$ under homogeneous boundary conditions on O_ε. We define

$$\Phi := \begin{cases} \varphi_o^\varepsilon, & \text{in } \overline{\Omega} \cap O_{\varepsilon/2}, \\ \Psi, & \text{in } \Omega_{\varepsilon/2} := \Omega \setminus (\overline{\Omega} \cap O_{\varepsilon/2}), \end{cases}$$

where $\Psi \geq \tau > 0$ is any regular enough function such that $\Phi \in C^2(\overline{\Omega})$. We are looking for the existence of a constant $K > 0$, large enough, such that function

$$\overline{u} := K\Phi$$

is a strict positive supersolution to problem (1).

We estimate now K on $\overline{\Omega} \cap O_{\varepsilon/2}$. Keeping in mind that $a(x) \geq 0$ and $K\Phi = K\varphi_o^\varepsilon$, it must be

$$\begin{cases} K(-\Delta\varphi_o^\varepsilon) \geq \lambda m K\varphi_o^\varepsilon - af(x, K\varphi_o^\varepsilon), & \text{in } \overline{\Omega} \cap O_{\varepsilon/2}, \\ K\varphi_o^\varepsilon \geq g > 0, & \text{on } \partial\overline{\Omega} \cap O_{\varepsilon/2} = \partial\Omega. \end{cases}$$

In the interior of the domain it should happen

$$K(-\Delta\varphi_o^\varepsilon) \geq \lambda m K\varphi_o^\varepsilon - af(x, K\varphi_o^\varepsilon),$$

which is equivalent to

$$K\sigma_\varepsilon\varphi_o^\varepsilon \geq \lambda m K\varphi_o^\varepsilon - af(x, K\varphi_o^\varepsilon)$$

by the definition of φ_o^ε. Rearranging terms we get

$$af(x, K\varphi_o^\varepsilon) \geq K\varphi_o^\varepsilon(\lambda m - \sigma_\varepsilon), \tag{5}$$

where the left hand side is non negative and the right one is strictly negative because of the selection of ε. Therefore, (5) is a proper inequality $\forall K > 0$. By construction, on the boundary we have $\varphi_o^\varepsilon|_{\partial\Omega}$, $g > 0$ and therefore, whenever

$$K > \max_{x \in \partial\Omega} \left\{ \frac{g(x)}{\varphi_o^\varepsilon(x)} \right\}, \tag{6}$$

the boundary condition is satisfied (with strict inequality).

In the domain $\Omega_{\varepsilon/2}$ it should be verified

$$K(-\Delta\Psi) \geq \lambda m K\Psi - af(x, K\Psi) \tag{7}$$

rearranging terms, expression (7) is equivalent to

$$\frac{f(x, K\Psi)}{K} \geq \max_{x \in \overline{\Omega}_{\varepsilon/2}} \left\{ \frac{\Delta\Psi + \lambda m\Psi}{a} \right\}. \tag{8}$$

As $a(x) > 0\ \forall x \in \Omega_{\varepsilon/2}$, function $\dfrac{\Delta\Psi + \lambda m\Psi}{a}$ reaches its maximum (provided a regular Ψ). By hypothesis on *f*, there exists K_0 such that $\forall K > K_0$ condition (8) holds. Thus, there exists $K > 0$ such that function $K\Phi$ is a strict positive supersolution to (1). Function $\underline{u} := 0$ is a subsolution to problem (1), and $\underline{u}(x) < \overline{u}(x)\ \forall x \in \overline{\Omega}$. □

Corollary 2.2. *For each* $\lambda \in \mathbb{R}$, *problem* (1) *has a unique positive solution*.

Proof. Since we have an ordered pair formed by a subsolution and a supersolution, a theorem due to Amann [1] guaranties the existence of a solution u to problem (1) such that $0 \leq u \leq K\Psi$. Uniqueness follows, for instance, from [11]. □

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